Problem I.4 ... curved space

7 points

Two point masses, each of mass m, are located on the surface of a sphere of radius R at an initial distance d_0 (where $R \gg d_0$). The segment connecting these points forms an arc on a great circle of the sphere, which we will refer to as the "equator" for the purposes of this problem. Both points are given the same initial velocity v directed "north" (i.e., perpendicular to the defined equator). No external forces act on the points, other than the constraint that keeps them on the surface of the sphere. The points move along great circles (meridians), which converge toward the pole. Consequently, the mutual distance between them decreases. Imagine an observer who is unaware that the points are moving on a curved spherical surface and assumes they are moving in a flat plane. This observer attributes the fact that the points are approaching each other to the existence of an apparent attractive force acting between them.

What dependence of this attractive force on the instantaneous distance d between the points would this observer derive from the experiment?

Lego was thinking about visualizations of relativity.

The problem could be solved using formulas from geodesy[1], but that would be unnecessarily complicated. Instead, we can use the assumption $d_0 \ll R$, which means that from the perspective of the points, the curvature of the surface beneath them is negligible. Moreover, the points will only approach each other, so at any later time their mutual distance d will also satisfy $d \ll R$.

When the points start moving perpendicular to the "equator", and no force acts on them (apart from the force that keeps them on the surface), they will move along "meridians". The difference in their "longitudes" will therefore remain constant. It then follows from the definition of an angle that the ratio of their distance to the circumference of the circle of latitude on which they are currently located will be constant.[2] In other words, their distance is proportional to the circumference of the circle of latitude on which they are located.

How does the circumference of a circle of latitude depend on the "latitude" θ? Imagine the following right triangle. We connect a point on the surface of the Earth to the center of the Earth, thereby forming the first side (the hypotenuse). We obtain the second side by dropping a perpendicular from the point on the surface to the Earth's axis. The third side connects the foot of this perpendicular to the center of the Earth along the Earth's axis. The hypotenuse has length R, and we denote the angle of the triangle at the point on the Earth's surface by θ; then the leg adjacent to this point has length $R\cos(\theta)$. This leg is also the radius of the corresponding circle of latitude. The circumference of the circle of latitude is therefore proportional to $\cos(\theta)$; the mutual distance of the two point masses then depends on the "latitude" θ as

$$d = d_0 \cos(\theta)\,,$$

while the "latitude" depends on the distance traveled s (from the definition of an angle) as

$$\theta = \frac{s}{R}\,.$$

[1] `https://en.wikipedia.org/wiki/Great-circle_distance`

[2] It is precisely here that we use the negligible curvature of the surface. Imagine points near the pole that are simultaneously located on meridians differing by 180°. Their shortest connection along the surface passes through the pole, not along a circle of latitude. At the same time, however, they would satisfy $d_0 = \pi R$. For points satisfying $d_0 \ll R$, such problems do not arise; you can either accept this or solve the problem exactly and verify that after using a Taylor expansion for $d_0 \ll R$, you obtain a result identical to ours.

Since only the constraint force actually acts on the points, no work is done on them, and their energy is therefore conserved. Consequently, their speed is also conserved, and $s = vt$. Substituting back, we obtain the time dependence of the distance between the points

$$d(t) = d_0 \cos\left(\frac{vt}{R}\right) . \tag{1}$$

The observer will determine the force from the relation $F = ma$. They move to a frame traveling at speed v together with the points and calculate the acceleration of the points relative to each other as the second time derivative of their mutual distance d. It remains to note that the same force acts on both points, so they have the same acceleration. Their relative acceleration is therefore twice the acceleration of either one of them: $a_v = 2a$. For the force F, we obtain

$$\begin{aligned} F = ma = \frac{1}{2}ma_v = \frac{1}{2}m\ddot{d} = \frac{1}{2}m\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\mathrm{d}}{\mathrm{d}t}\left(d_0\cos\left(\frac{vt}{R}\right)\right)\right) = \frac{1}{2}md_0\frac{\mathrm{d}}{\mathrm{d}t}\left(-\frac{v}{R}\sin\left(\frac{vt}{R}\right)\right) \\ = -\frac{1}{2}md_0\left(\frac{v}{R}\right)^2\cos\left(\frac{vt}{R}\right), \end{aligned}$$

where the minus sign indicates that the force is attractive. Substituting from equation (1), we convert the time dependence of the force into a dependence on the distance d; we obtain

$$F = -\frac{1}{2}md\left(\frac{v}{R}\right)^2 .$$

The observer may therefore conclude that the attractive force acting between the points is proportional to their distance. From further experiments, it would also be possible to conclude, among other things, that this force is always proportional to the mass m. This is related to why Einstein concluded that the gravitational force is in fact an inertial force, and why this theory can be represented in terms of the curvature of spacetime.

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