Problem VI.4 ... hit and sunk! 7 points

A torpedo created a hole of surface area A in the hull of a submarine. The hull has a volume V. Will the submarine reach the surface if hit at depth h_0? It can ascend toward the surface with speed v, and the pumps are able to remove a volume of V_1 each second; the critical volume after which the submarine inevitably sinks to the bottom, where it is crushed by the surrounding water pressure, is equal to $V_{\mathrm{crit.}} \ll V$. Assume that v_{v}, the speed at which the water is flowing into the submarine, obeys Torricelli's law regardless of the water volume inside the submarine.

Viktor was pondering upon what it feels like to be down and out.

Let h be the depth at which the submarine is located at time t. For h, the following equation holds:

$$h = h_0 - vt\,.$$

Using Torricelli's formula, we can determine the speed v_{v} at which water is flowing into the submarine as

$$\frac{1}{2}\rho v_{\mathrm{v}}^2 = \rho h g\,,$$

$$v_{\mathrm{v}} = \sqrt{2hg}\,,$$

where ρ is the water density and g the gravitational acceleration. Let V_2 be the volume of water that flows into the submarine every second; it is evident that

$$V_2 = v_{\mathrm{v}} S\,,$$

$$V_2 = \sqrt{2hg} S$$

is true.

The speed of water entering the submarine is gradually decreasing; therefore, at the beginning, the amount of water entering the submarine is greater than the amount the pumps are able to pump out and the volume of water is increasing. At some point after the torpedo hit the submarine, the amount of water flowing in and pumped out will equalize; after this moment, the total volume of water in the submarine will only decrease, that is, if the volume at that point is lesser than $V_{\mathrm{crit.}}$. Let T be the time between the torpedo impact and the moment when the volume of water inside the submarine reaches its maximum.[1] During this time, it is clear that $V_1 = V_2$. From this and from the monotonicity of $V(t)$ on the interval $t \in \{\tau | V_1 < V_2(\tau)\}$ (see discussion above), we can deduce that the submarine will not sink if

$$V(T) < V_{crit.}$$

holds.

For time T, we can calculate

$$V_1 = \sqrt{2hg} S\,,$$

$$V_1 = \sqrt{2g\left(h_0 - vT\right)} S\,,$$

$$T = \frac{h_0}{v} - \frac{1}{2vg}\left(\frac{V_1}{S}\right)^2\,.$$

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[1] It is very important to set the condition for T as we did here. If we chose the time of resurfacing instead and continued in the same way this solution does, we would integrate over the negative volume of water that is being pumped out of the submarine. If we did that, we would obtain values lower than $V_{\mathrm{crit.}}$. However, when "overcoming" the maximum, the submarine would still sink due to the behaviors described in the problem statement.

The volume $V(T)$ can then be obtained by integration as

$$V(T) = \int_0^T \left(\sqrt{2g(h_0 - vt)}S - V_1\right) \mathrm{d}t \,.$$

The solution to this integral is

$$V(T) = \left[-\frac{2\sqrt{2g}S}{3v}\left(h_0 - vt\right)^{\frac{3}{2}} - V_1 t\right]_0^T \,,$$

after substituting for the lower and upper bound of the integral

$$V(T) = \frac{2\sqrt{2g}S}{3v}\left[h_0^{\frac{3}{2}} - \left(\frac{1}{2g}\right)^{\frac{3}{2}}\left(\frac{V_1}{S}\right)^3\right] - \frac{V_1}{v}\left[h_0 - \frac{1}{2g}\left(\frac{V_1}{S}\right)^2\right] \,.$$

As per the discussion in the beginning, the submarine will not sink if the formula above is of lesser value than $V_{\mathrm{crit.}}$.

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