Problem VI.1 ... broken swing

3 points

Martin attached a weight to a swing with a suspension length $l = 2\,\mathrm{m}$ and released it from the horizontal position. What is the mass m of the attached weight if the massless suspension, with a maximum tension $T_{\mathrm{max}} = 1\,\mathrm{kN}$, broke at the moment when it formed an angle $\varphi = 20^\circ$ with the vertical? Assume that the only object with mass is the weight itself, and that the suspension remains taut at all times.

Martin was swinging, and then he wasn't.

For the suspension to break, a force greater than its load capacity must act on it. A centrifugal force and a gravitational force act on the weight of mass m; the centrifugal force $\mathbf{F}_\mathrm{o}$ acts in the direction of the rope extension, while for the gravitational force $\mathbf{F}_G$, we are interested only in the contribution in the radial direction (obtained by resolving the force into components using sine and cosine). Formally, the rope therefore breaks when

$$T_{\mathrm{max}} = F_{G,r} + F_\mathrm{o} = mg\cos\varphi + ma_\mathrm{d}\,, \tag{1}$$

where m is the mentioned mass of the weight, g is the gravitational acceleration, φ is the deflection angle, and $a_\mathrm{d} = v^2/l$ is the centripetal acceleration for uniform motion along a circle.

To determine the speed of this motion, we use the law of conservation of energy; our weight has contributions from kinetic energy E_k and potential energy E_p.

$$E_{\mathrm{total}} = E_\mathrm{p} + E_\mathrm{k} = mgh + \frac{1}{2}mv^2.$$

Let us choose the lowest point that the swing can reach as the zero level of potential energy. Relative to this level, the suspension (and the weight at $\varphi = 90^\circ$, the maximum deflection given in the problem statement) is at height l. At the maximum deflection, the speed (and therefore also the kinetic energy) is zero, so the total energy is equal to $E_{\mathrm{total}} = mgl$. It remains to express h as a function of l so that we can use the known quantities; clearly, however, this is again only a resolution into components. We need the vertical component, so

$$h = l(1-\cos\varphi)\,,$$

which allows us to express the square of the speed v as

$$\begin{aligned} E_\mathrm{k} &= E_{\mathrm{total}} - E_\mathrm{p}\,,\\ \frac{1}{2}mv^2 &= mgl - mgl(1-\cos\varphi)\,,\\ v^2 &= 2gl\cos\varphi\,. \end{aligned}$$

Now we can easily express the mass m from equation (1) and substitute the derived speed together with the given values into the form

$$T_{\max} = m\left(g\cos\varphi + \frac{v^2}{l}\right) = m\left(g\cos\varphi + 2g\cos\varphi\right),$$

$$m = \frac{T_{\max}}{3g\cos\varphi} \doteq 36.16\,\mathrm{kg}\,.$$

Therefore, we attached a weight of mass 36.16 kg to the swing.

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