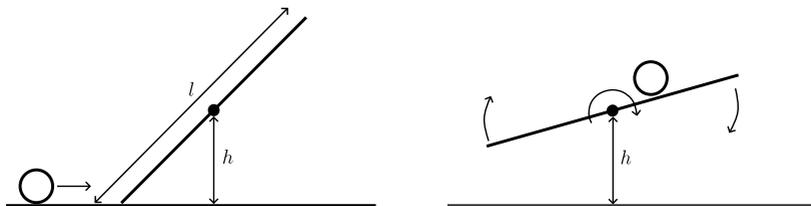


Problem V.4 . . . minigolf see-saw

8 points

Maybe you have gone minigolfing at a course where one of the holes featured a see-saw obstacle. We can model the see-saw as a board of mass M and length l . This board can rotate around a horizontal axis passing through the center of the board; the axis is located at height h above the ground. A golf ball of mass m and radius r rolls toward the see-saw from a direction perpendicular to the axis of rotation, moves onto the see-saw, and gradually tips the board, allowing the ball to descend on the other side. The ball rolls without slipping throughout the process. Formulate the equations of motion for this system—you do not have to solve them.



Fykosaurus scored a birdie on the last hole and won.

This problem can be solved in several ways. We will show the approach that is similar to high school methods. Before we start solving the problem, let us go over what happens to the ball while it is passing over the seesaw.

At first, the ball will just slow down with a constant negative acceleration. When the ball passes over the middle of the ramp, the seesaw will start to overturn due to the ball's weight, and the slope of the inclined plane will change. This will also cause the ball's acceleration to change, which in turn will affect the speed of the seesaw's rotation. We can see that the system will exhibit some interesting dynamics. Let's make one more important observation: the ball and the seesaw will be in contact throughout their passage.

We can see this with the following reasoning: since at first both objects will be touching and will not be rotating, it would be necessary for them to accelerate at a different rate to separate them. But that will not happen because the points of the seesaw will accelerate precisely because of the ball's gravity, and if the ball is not in contact, the seesaw will not accelerate, and the ball will "catch up".

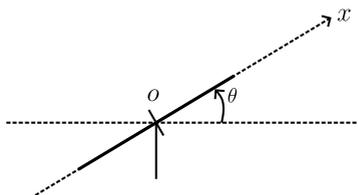


Figure 1: Chosen reference system.

Now we can introduce a coordinate system as in figure 1. We place the origin of said coordinate system at the center of gravity of the rotating plate—the coordinate x determines the distance between the ball and the center of the seesaw—the positive direction is the one the ball initially moved in, and θ denotes the angle of rotation of the seesaw.

We will divide the process into two phases: (1) the movement of the ball up to the center of the seesaw in a case where it has enough energy to do so; (2) the movement of the ball and the seesaw after it gets past the center. The first case is a simple application of the energy conservation principle, since the seesaw remains stable and only the ball is moving. We will now calculate the amount of energy the ball enters the first part of process. We will set the potential energy level to zero; therefore all is left is the kinetic energy, which consists of translational energy and rotational energy. For a sphere of radius r , the moment of inertia is defined as $J = (2/5)mr^2$. Using the formula $\omega = v/r$, where ω is the angular velocity of the sphere, v its velocity, we obtain the relation for kinetic energy

$$E_k = \frac{1}{2}mv^2 + \frac{1}{2}J\omega^2 = \frac{7}{10}mv^2.$$

As for the energy when moving on the seesaw, we have derived the relation for kinetic energy. To calculate the potential energy, we will use trigonometric relations, to find the change in height using our coordinates. It is important to remember that θ_0 is constant, since when the seesaw starts to rotate, the height of its center of gravity changes “above the seesaw” due to the size of the ball, but more on that later. We will thus express the potential energy as a term

$$E_p = (l/2 + x) \sin(\theta_0)mg,$$

where θ_0 is the initial incline angle of the seesaw; we can express it as $\theta_0 = \arcsin(2h/l)$. Now, we are able to describe the motion of the ball by the equation

$$(l/2 + x) \sin(\theta_0)mg + \frac{7}{10}m\dot{x}^2 = \frac{7}{10}mv_0^2.$$

Here, the dot above the variable indicates its time derivative, in other words, its change in time. In the case of the x coordinate, this indicates its velocity. Since this is a differential equation (an equation with a derivative), it is good to determine the initial conditions, namely $x(0) = -l/2$ and $\dot{x}(0) = v_0$.

We will now move on to the second phase, which occurs if $(7/5)mv^2 > l \sin(\theta_0)mg$ holds. We derive the first equation from the movement of the ball in the direction of the coordinate x , i.e., the radial direction. In the inertial frame of reference, the ball is primarily affected by two radial forces: weight $F_g = mg \sin(\theta)$ and the frictional force, hereby denoted as F_T . Since the frame of reference in relation to which we express the movement of the ball, i.e., the x axis, rotates with respect to the inertial frame of reference, fictitious forces act on the object as well. The resulting force acting on the ball in the direction of the x axis in the non-inertial rotating frame of reference F_N is described by the equation

$$F_N = F_T - 2m\dot{\theta} \times \dot{x} - m\ddot{\theta} \times (\dot{\theta} \times x) - m\ddot{\theta} \times x.$$

However, we are only interested in the components of the forces parallel to x . After substituting into the vector products, we decompose the resulting fictitious forces into the direction parallel to x and perpendicular to x using trigonometric identities in a right-angled triangle with sides r , x and $r_k = \sqrt{x^2 + r^2}$, the position vector of the center of gravity of the ball. We get the result

$$F_N = -mg \sin \theta - F_T + m\dot{\theta}^2 x - m\ddot{\theta} r,$$

where the minus sign for both real forces acting in the inertial frame of reference is given by the fact that for positive θ , both real forces act against the direction of the ball’s movement.

Notice that the Coriolis force completely disappears as it acts perpendicularly to x and that the term $m\ddot{\theta}r$ —the component of the so-called Euler force, which arises as a result of the acceleration of the rotation of the frame of reference—from the notation we can notice that its direction in the direction x is given by the radius of the ball, so for a mass point this term would disappear and we would get a typical relation for the rotation of a mass point around an external axis, which is influenced only by inertial forces. All that remains is to substitute for F_T , which in an inertial frame of reference causes the ball to rotate according to the formula

$$F_T r = \frac{2}{5} m r^2 \varepsilon,$$

where ε is the angular acceleration of the ball, which we can rewrite from the condition of perfect rolling to $r\varepsilon = \ddot{x}$. Combining all the above equations, we get the result

$$\frac{7}{5} m \ddot{x} = -mg \sin \theta + m \dot{\theta}^2 x - m \ddot{\theta} r. \quad (1)$$

This gives us the first equation connecting the two unknowns x, θ . For a complete description of the motion of this system, we need to obtain a second constraint, which we obtain if we consider the second momentum theorem

$$\frac{d\mathbf{L}}{dt} = \mathbf{M}_F,$$

where \mathbf{L} is the angular momentum of the ball's center of mass around the origin of our coordinate system and \mathbf{M}_F is the moment of all external forces acting on the ball's center of mass. We express \mathbf{L} first as

$$\mathbf{L} = \mathbf{r} \times \mathbf{p} + J\boldsymbol{\omega},$$

where \mathbf{r} is the position vector of the ball's center of mass, \mathbf{p} is the momentum of the ball's center of mass, $J = (2/5)mr^2$ is the moment of inertia of the ball and $\boldsymbol{\omega} = \dot{x}/r$ is its angular velocity. The velocity of the ball's center of mass is given by the ball's own motion on the seesaw and the seesaw's rotation—the ball rotates around the origin of the coordinate system with the same angular velocity as the seesaw—see the discussion at the beginning. We can therefore write

$$L = m(r^2 + x^2)\dot{\theta} + mr\dot{x} + \frac{2}{5}mr\dot{x},$$

where we have already written L as a scalar; the vector L would point to the drawing board in the drawing above. We will now express the total moment of external forces that affect the motion of the center of gravity.

$$\mathbf{M}_F = \mathbf{r} \times \mathbf{F}.$$

The forces acting on the center of gravity are both components of the gravitational force and the force T the seesaw exerts on the ball (which is generally different from $mg \cos \theta$ due to the accelerated motion of the seesaw).

$$M_F = x(mg \cos \theta - T) - rmg \sin \theta,$$

where we have already written M as a scalar. The signs are chose to align vectors \mathbf{M} and \mathbf{L} . The same equation would describe the resulting moment of forces on a stationary inclined plane, or around the base of the inclined plane. In that case, the force exerted on the seesaw would be the same as the force exerted by the seesaw on the ball $mg \cos \theta = T$, which would cancel

out this term. In the case of a stationary seesaw, the term containing $\dot{\theta}$ would disappear. By simply differentiating L with respect to time, we obtain the second equation of motion

$$mr^2\ddot{\theta} + 2mx\dot{x}\dot{\theta} + mx^2\ddot{\theta} + \frac{7}{5}mr\dot{x} = mxg \cos \theta - Tx - rmg \sin \theta.$$

However, the equation still contains the term Tx , which we have not yet found out how to express it. We can determine this term from the rotation of the seesaw; according to Newton's third law, the force Tx also acts on the seesaw. It is also the only force acting on the seesaw, and it is therefore causing the rotation of the seesaw, for that reason

$$Tx = J_h \ddot{\theta},$$

where $J_h = (1/12)Ml^2$ denotes the moment of inertia of the seesaw around the origin of our chosen coordinate system.

The second equation of motion then takes the form

$$\frac{1}{12}Ml^2\ddot{\theta} + mr^2\ddot{\theta} + 2mx\dot{x}\dot{\theta} + mx^2\ddot{\theta} + \frac{7}{5}mr\dot{x} = mxg \cos \theta - rmg \sin \theta. \quad (2)$$

Equations (1) and (2) are two differential equations for two unknowns, x and θ , so they are equations of motion of the system. Second option, how we can derive the second equation of motion is to look at the situation again in a non-inertial frame of reference, however, this time, work out the forces perpendicular to the seesaw. From the formula for converting forces acting in an inertial frame of reference to a non-inertial one, considering only those in the direction perpendicular to x , we get

$$F_N^k = -mg \cos \theta + T + 2m\dot{\theta}\dot{x} + m\ddot{\theta}x + m(\dot{\theta})^2r.$$

Since the ball does not come off the seesaw (or pass through it), in a non-inertial frame of reference the force F_N^k acting on the ball perpendicular to x must be zero. This means that we get the equation

$$mg \cos \theta = T + 2m\dot{\theta}\dot{x} + m\ddot{\theta}x + m(\dot{\theta})^2r. \quad (3)$$

However, this does not match the previous equation, does this mean there is a problem? Well, let us try to play with this equation a bit more before we make a verdict. Notice that, compared to the other terms in the equation, there is an additional term, $m(\dot{\theta})^2r$, and we can also find a similar term in the equation (1), so let us try to express it from there as

$$m\dot{\theta}^2xr = \frac{7}{5}mr\dot{x} + mgr \sin \theta + m\ddot{\theta}r^2,$$

If we multiply the equation (3) by x , substitute the previous term and express Tx using the same reasoning as above, we get

$$mgx \cos \theta = J_h \ddot{\theta} + 2mx\dot{\theta}\dot{x} + m\ddot{\theta}x^2 + \frac{7}{5}mr\dot{x}x + mgr \sin \theta + m\ddot{\theta}r^2x,$$

which, after rearranging the terms and substituting for J_h , is the same equation as equation (2).

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