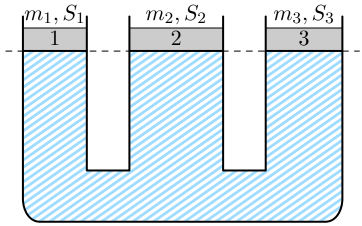
## Problem V.3 ... hydraulic W

6 points

*Consider a hydraulic press using three pistons with cross-sectional areas $S_1$, $S_2$, $S_3$ and masses $m_1$, $m_2$, $m_3$. Initially, they are at the same height. What is the acceleration of piston 1 at the moment when we release all of them simultaneously?*




*Lego remembered a problem that had once been assigned to elementary school students.*

Since all pistons are located at the same height, the pressure differences at their locations due to hydrostatic pressure will be zero. Therefore, at each of them there is water at the same pressure; let us denote this pressure as $p$. Then a pressure force $F_{p1} = pS_1$ acts on the first piston vertically upward, and analogously for the remaining two pistons.

The gravitational force acting on the pistons is $F_{gi} = m_i g$ downward. Thus, the equations of motion for the pistons are

$$a_1 m_1 = m_1 g - pS_1 \,,$$
$$a_2 m_2 = m_2 g - pS_2 \,,$$
$$a_3 m_3 = m_3 g - pS_3 \,,$$

where the accelerations $a_i$ are unknown, as is the pressure $p$. We therefore currently have three equations and four variables we want to solve for.

The last required equation is obtained from the fact that the volume of the hydraulic fluid must be conserved. Mathematically, we can express this condition as

$$h_1 S_1 + h_2 S_2 + h_3 S_3 = V = \mathrm{const}\,.$$

where $h_i$ denotes the height of the $i$-th piston. We then either use intuition, or differentiate this relation twice with respect to time to obtain

$$a_1 S_1 + a_2 S_2 + a_3 S_3 = 0\,,$$

which is the equation we needed in order to have 4 equations for 4 variables. It remains to solve the resulting system.

We divide each of the equations of motion by the corresponding $S_i$

$$a_1 \frac{m_1}{S_1} = g\frac{m_1}{S_1} - p\,,$$
$$a_2 \frac{m_2}{S_2} = g\frac{m_2}{S_2} - p\,,$$
$$a_3 \frac{m_3}{S_3} = g\frac{m_3}{S_3} - p\,,$$

We then eliminate the unknown pressure $p$ by subtracting these equations; for example, we subtract the 1st equation from the 2nd and the 1st from the 3rd.

$$a_2 \frac{m_2}{S_2} - a_1 \frac{m_1}{S_1} = g \left( \frac{m_2}{S_2} - \frac{m_1}{S_1} \right) ,$$
$$a_3 \frac{m_3}{S_3} - a_1 \frac{m_1}{S_1} = g \left( \frac{m_3}{S_3} - \frac{m_1}{S_1} \right) .$$

From the condition of volume conservation, we express $a_3$

$$-\frac{S_1}{S_3} a_1 - \frac{S_2}{S_3} a_2 = a_3 ,$$

and substitute it into the second equation

$$a_2 \frac{m_2}{S_2} - a_1 \frac{m_1}{S_1} = g \left( \frac{m_2}{S_2} - \frac{m_1}{S_1} \right)$$
$$-a_1 \frac{S_1}{S_3} \frac{m_3}{S_3} - a_2 \frac{S_2}{S_3} \frac{m_3}{S_3} - a_1 \frac{m_1}{S_1} = g \left( \frac{m_3}{S_3} - \frac{m_1}{S_1} \right) .$$

It remains to eliminate $a_2$, so we express it in both equations

$$\begin{aligned} a_2 &= a_1 \frac{S_2}{m_2} \frac{m_1}{S_1} + \frac{S_2}{m_2} g \left( \frac{m_2}{S_2} - \frac{m_1}{S_1} \right) \\ &= -a_1 \frac{S_3}{S_2} \frac{S_3}{m_3} \left( \frac{m_1}{S_1} + \frac{S_1}{S_3} \frac{m_3}{S_3} \right) - g \frac{S_3}{S_2} \frac{S_3}{m_3} \left( \frac{m_3}{S_3} - \frac{m_1}{S_1} \right) , \end{aligned}$$

and therefore their right-hand sides must be equal

$$-a_1 \frac{S_3}{S_2} \frac{S_3}{m_3} \left( \frac{m_1}{S_1} + \frac{S_1}{S_3} \frac{m_3}{S_3} \right) - g \frac{S_3}{S_2} \frac{S_3}{m_3} \left( \frac{m_3}{S_3} - \frac{m_1}{S_1} \right) = a_1 \frac{S_2}{m_2} \frac{m_1}{S_1} + \frac{S_2}{m_2} g \left( \frac{m_2}{S_2} - \frac{m_1}{S_1} \right) ,$$
$$g \left( \frac{S_3}{S_2} \frac{S_3}{m_3} \frac{m_1}{S_1} - \frac{S_3}{S_2} \frac{S_3}{m_3} \frac{m_3}{S_3} - \frac{S_2}{m_2} \frac{m_2}{S_2} + \frac{S_2}{m_2} \frac{m_1}{S_1} \right) = a_1 \left( \frac{S_3}{S_2} \frac{S_3}{m_3} \frac{m_1}{S_1} + \frac{S_3}{S_2} \frac{S_3}{m_3} \frac{S_1}{S_3} \frac{m_3}{S_3} + \frac{S_2}{m_2} \frac{m_1}{S_1} \right) ,$$
$$g \left( \frac{S_3}{S_2} \frac{S_3}{m_3} \frac{m_1}{S_1} - \frac{S_3}{S_2} - 1 + \frac{S_2}{m_2} \frac{m_1}{S_1} \right) = a_1 \left( \frac{S_3}{S_2} \frac{S_3}{m_3} \frac{m_1}{S_1} + \frac{S_1}{S_2} + \frac{S_2}{m_2} \frac{m_1}{S_1} \right) ,$$
$$g \frac{\frac{S_3}{S_2} \frac{S_3}{m_3} \frac{m_1}{S_1} + \frac{S_2}{m_2} \frac{m_1}{S_1} + \frac{S_1}{S_2} - \frac{S_1}{S_2} - \frac{S_3}{S_2} - 1}{\frac{S_3}{S_2} \frac{S_3}{m_3} \frac{m_1}{S_1} + \frac{S_1}{S_2} + \frac{S_2}{m_2} \frac{m_1}{S_1}} = a_1 ,$$
$$g - g \frac{\frac{S_1}{S_2} + \frac{S_3}{S_2} + 1}{\frac{S_3}{S_2} \frac{S_3}{m_3} \frac{m_1}{S_1} + \frac{S_1}{S_2} + \frac{S_2}{m_2} \frac{m_1}{S_1}} = a_1 ,$$
$$g - g \frac{m_2 m_3 S_1 (S_1 + S_2 + S_3)}{m_1 m_2 S_3^2 + m_2 m_3 S_1^2 + m_3 m_1 S_2^2} = a_1 .$$

The final result makes physical sense: the piston will descend with an acceleration smaller than $g$; the amount by which this acceleration is reduced increases with both $m_2$ and $m_3$. If these masses are sufficiently large, we may even obtain $a_1 < 0$, which would mean that piston 1 moves upward.

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