Problem IV.5 ... pulleys and charges 9 points

Two weights are suspended from a pulley. One weight is of mass m_1 and the other of mass m_2. The mass m_1 carries an electric charge Q_1, while the mass m_2 is not charged. A point electric charge Q is located above the pulley or below the pulley at a distance h. Is there an equilibrium position for the weights on the pulley? Is it stable? Consider the radius of the pulley to be negligible. Solve the problem for the following cases:

1. *The charge $Q = -1.00 \cdot 10^{-5}\,\mathrm{C}$ is located at a height $h = 0.15\,\mathrm{m}$ above the pulley. The values of the weights are $m_1 = 1.00\,\mathrm{kg}$, $m_2 = 2.00\,\mathrm{kg}$, $Q_1 = -1.00 \cdot 10^{-5}\,\mathrm{C}$. The length of the rope connecting the two weights via the pulley is $l = 1.00\,\mathrm{m}$.*
2. *The charge $Q = 5.00 \cdot 10^{-6}\,\mathrm{C}$ is located at a height $h = 0.30\,\mathrm{m}$ above the pulley. The values of the weights are $m_1 = 1.50\,\mathrm{kg}$, $m_2 = 2.00\,\mathrm{kg}$, $Q_1 = 2.00 \cdot 10^{-4}\,\mathrm{C}$. The length of the rope connecting the two weights via the pulley is $l = 0.75\,\mathrm{m}$.*
3. *The charge $Q = 4.00 \cdot 10^{-5}\,\mathrm{C}$ is located at a depth of $h = 0.50\,\mathrm{m}$ below the pulley. The values of the weights are $m_1 = 1.20\,\mathrm{kg}$, $m_2 = 1.90\,\mathrm{kg}$, $Q_1 = -2.00 \cdot 10^{-5}\,\mathrm{C}$. The length of the rope connecting the two weights is $l = 2.00\,\mathrm{m}$.*

The weights can only move in the vertical direction.

Jindra read a book "Lego's Selected Problems on Pulleys I".

We can solve the problem most conveniently using potential energy. First, let us look at the first two subproblems, where the charge Q was located above the pulley.

If the weight m_1 is at a depth x below the pulley, then the weight m_2 is at a depth $l - x$ below the pulley, where l is the length of the rope connecting the two weights. For the rope to remain guided over the pulley, we must have $x < l$ and $l - x < l$.

The gravitational potential energies of the weights m_1 and m_2 are

$$E_{g1} = -m_1 g x\,,$$
$$E_{g2} = -m_2 g(l - x)\,.$$

The electric potential energy of the weight m_1 is

$$E_{e1} = \frac{1}{4\pi\varepsilon_0}\frac{QQ_1}{h + x}\,.$$

Based on the definition of our coordinates and the constraints imposed by the pulley, the value of x is necessarily positive, as is $h + x$. The total potential energy of the system as a function of the position of the weight m_1 is therefore

$$E = -m_1 g x + m_2 g(x - l) + \frac{1}{4\pi\varepsilon_0}\frac{QQ_1}{h + x}\,.$$

We differentiate this expression with respect to x and set the derivative equal to zero to find the equilibrium points:

$$\frac{\mathrm{d}E}{\mathrm{d}x} = -m_1 g + m_2 g - \frac{1}{4\pi\varepsilon_0}\frac{QQ_1}{(h + x)^2} = 0\,,$$

$$(h + x)^2 = \frac{QQ_1}{4\pi\varepsilon_0 g(m_2 - m_1)}\,,$$

$$x = \sqrt{\frac{QQ_1}{4\pi\varepsilon_0 g(m_2 - m_1)}} - h\,.$$

We see that a solution can exist only if the terms QQ_1 and $(m_2 - m_1)$ have the same sign. To determine whether this is a local minimum or maximum of the potential energy, we compute the second derivative and substitute the coordinate of the equilibrium point found:

$$\frac{\mathrm{d}^2 E}{\mathrm{d}x^2} = \frac{1}{2\pi\varepsilon_0} \frac{QQ_1}{(h+x)^3} > 0\,.$$

The second derivative of the function is positive at the equilibrium point, so this is a local minimum of the potential energy, and the position is stable.

In subproblem 1, after substituting the values from the problem statement, we obtain the result $x = 0.15\,\mathrm{m}$. This position is also stable. The second weight m_2 would be located at a depth $l - x = 0.85\,\mathrm{m}$ below the pulley.

In subproblem 2, the numerical result is $x = 1.05\,\mathrm{m}$. However, this does not satisfy the necessary condition $x < l$, and therefore, no equilibrium position exists in this case.

Now let us look at subproblem 3, where the charge Q was located at a depth h below the pulley. We will again measure the position x of the weight m_1 downward from the pulley. The gravitational potential energies of the weights m_1 and m_2 are

$$\begin{aligned} E_{g1} &= -m_1 g x\,, \\ E_{g2} &= -m_2 g (l - x)\,. \end{aligned}$$

The electric potential energy of the weight m_1 is

$$E_{e1} = \frac{1}{4\pi\varepsilon_0} \frac{QQ_1}{|h - x|}\,.$$

Here, we must divide by the absolute value of the difference, because the potential-energy field is spherically symmetric around the point charge. The electrostatic potential energy cannot change sign when we move from a position above the charge to a position below the charge. The total potential energy of the system as a function of the position of the weight m_1 is therefore

$$E = -m_1 g x - m_2 g (l - x) + \frac{1}{4\pi\varepsilon_0} \frac{QQ_1}{|h - x|}\,.$$

Based on the definition of our coordinates and the constraints imposed by the pulley, the value of x is again necessarily positive; however, the value of $h - x$ can be either positive or negative. We again differentiate this expression with respect to x and set the derivative equal to zero to find the equilibrium points.

Let us first consider the solution branch for which we require $h - x > 0$. In this case, the weight m_1 is above the charge:

$$\begin{aligned} \frac{\mathrm{d}E}{\mathrm{d}x} &= -m_1 g + m_2 g + \frac{1}{4\pi\varepsilon_0} \frac{QQ_1}{(h-x)^2} = 0\,, \\ (h - x)^2 &= \frac{QQ_1}{4\pi\varepsilon_0 g (m_1 - m_2)}\,, \\ x &= h - \sqrt{\frac{QQ_1}{4\pi\varepsilon_0 g (m_1 - m_2)}}\,. \end{aligned}$$

We see that a solution exists only if the terms QQ_1 and $(m_1 - m_2)$ have the same sign. After substituting the numbers from subproblem 3, we get the result $x = -0.523\,\mathrm{m}$, but this does not make physical sense, since the weight m_1 would have to be above the pulley. Under the requirement $h - x > 0$, there is therefore no equilibrium position.

Now let us consider the solution branch where we require $h - x < 0$; the weight m_1 is then below the charge:

$$\frac{\mathrm{d}E}{\mathrm{d}x} = -m_1 g + m_2 g - \frac{1}{4\pi\varepsilon_0} \frac{QQ_1}{(x-h)^2} = 0\,,$$

$$(x-h)^2 = \frac{QQ_1}{4\pi\varepsilon_0 g(m_2 - m_1)}\,,$$

$$x = \sqrt{\frac{QQ_1}{4\pi\varepsilon_0 g(m_2 - m_1)}} + h\,.$$

In that case, an equilibrium position can exist only if the terms QQ_1 and $(m_2 - m_1)$ have the same sign. However, the numerical values in the problem statement are inconsistent with this requirement, and therefore, no equilibrium position exists here either.

You may have noticed that by differentiating the potential energy, we obtained the usual force-balance condition. Of course, we could have written the equation for the equilibrium of the gravitational and electrostatic forces directly, but we found it more illustrative to proceed through potential energy. In more complicated systems with more objects and in higher dimensions, it is usually more difficult to set up the force-balance equations than to set up the equations for the potential energy. Moreover, when differentiating the potential energy, there is less room for error than when thinking through all the forces. In addition, potential energy has broader applications than force; for example, we can use it in constructing the Lagrangian or Hamiltonian of the system under consideration.

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