Problem IV.2 ... partial overpressure

3 points

Consider two containers filled with an ideal gas of equal volume and temperature, consisting of a mixture of oxygen and nitrogen. Each container is sealed at the top by a piston of identical thickness and material, positioned at the same height. The pistons exert pressure on the gas only through their own weight and do not interact with the container walls.

The ratio of the partial pressures of oxygen in the first and second containers is $3:5$. *The partial pressure of nitrogen in the first container exceeds that in the second by* $40\,\mathrm{Pa}$, *and the sum of the oxygen partial pressures in both containers equals the sum of the nitrogen partial pressures. Determine the resulting displacement of the pistons and the total pressure after the containers will be interconnected.*

Monča is partially ripping off Mechanics I problem set.

First, we calculate the relation between the total pressures. We know that both vessels are closed by pistons of the same thickness h, made of the same material; thus, they have density ρ. Denote the piston cross-sectional areas by S_1, S_2, the weights of the pistons by F_1, F_2, and the gravitational acceleration by g. In this case, the gas pressure in each vessel (p_1 for the first vessel and p_2 for the second) can be written as:

$$p_1 = \frac{F_1}{S_1} = \frac{\rho g h S_1}{S_1} = \rho g h\,,$$
$$p_2 = \frac{F_2}{S_2} = \frac{\rho g h S_2}{S_2} = \rho g h\,,$$

from which we easily see that both pressures are actually equal, and after connecting the vessels, the total pressure p will also be the same:

$$p = \frac{F_1 + F_2}{S_1 + S_2} = \frac{\rho g h\,(S_1 + S_2)}{S_1 + S_2} = \rho g h\,.$$

Now we express everything using partial pressures. The partial pressures of a gas mixture are characterized by the fact that their sum gives the total pressure of the mixture. For the pressure in the first vessel, we have:

$$p_1 = p^1_{\mathrm{O}_2} + p^1_{N_2}\,,$$

and for the second vessel:

$$p_2 = p^2_{O_2} + p^2_{N_2}\,,$$

where the above variables correspond to the pressures of individual gases (distinguished by the subscript) in the vessels (denoted by the superscript); that is, $p^1_{O_2}$ is the partial pressure of oxygen in the first vessel, and similarly for the others. From the relations between the individual partial pressures given in the problem, we obtain:

$$\frac{p^1_{O_2}}{p^2_{O_2}} = \frac{3}{5} \qquad \Rightarrow \qquad p^1_{O_2} = \frac{3}{5} p^2_{O_2}$$
$$p^1_{N_2} = p^2_{N_2} + 40\,\mathrm{Pa}\,,$$
$$p^1_{O_2} + p^2_{O_2} = p^1_{N_2} + p^2_{N_2}\,.$$

Substituting these relations into the equality of pressures gives:

$$
\begin{aligned}
p_1 &= p_2\,,\\
p^1_{O_2} + p^1_{N_2} &= p^2_{O_2} + p^2_{N_2}\,,\\
\frac{3}{5}p^2_{O_2} + p^1_{N_2} &= p^2_{O_2} + p^2_{N_2}\,,\\
p^1_{N_2} - p^2_{N_2} &= \frac{2}{5}p^2_{O_2}\,,\\
40\,\mathrm{Pa} &= \frac{2}{5}p^2_{O_2}\,,\\
p^2_{O_2} &= 100\,\mathrm{Pa}\,.
\end{aligned}
$$

Now we compute the oxygen pressure in the first vessel from the ratio:

$$p^1_{O_2} = \frac{3}{5}p^2_{O_2} = \frac{3}{5}100\,\mathrm{Pa} = 60\,\mathrm{Pa}\,.$$

From the equality of the sums of partial pressures in both vessels, it follows:

$$p^1_{O_2} + p^2_{O_2} = 60\,\mathrm{Pa} + 100\,\mathrm{Pa} = 160\,\mathrm{Pa} = p^1_{N_2} + p^2_{N_2} = p^2_{N_2} + p^2_{N_2} + 40\,\mathrm{Pa}$$

$$2p^2_{N_2} = 120\,\mathrm{Pa} \quad\Rightarrow\quad p^2_{N_2} = 60\,\mathrm{Pa}\,.$$

Then it holds that

$$p^1_{N_2} = p^2_{N_2} + 40\,\mathrm{Pa} = 100\,\mathrm{Pa}$$

and the total pressure after connecting the vessels is, according to the derived relation $p = p_1 = = p_2$:

$$p_1 = p^1_{\mathrm{O}_2} + p^1_{N_2} = 160\,\mathrm{Pa}\,,\quad p_2 = p^2_{O_2} + p^2_{N_2} = 160\,\mathrm{Pa} \quad\Rightarrow\quad p = 160\,\mathrm{Pa}\,.$$

Finally, we determine the change in piston positions. Before connection, the gas in both vessels has the same volume ($V_1 = V_2 = V$), temperature ($T_1 = T_2 = T$), and also pressure ($p_1 = p_2 = p$). Let n be the amount of substance of the gas and R the molar gas constant. From the ideal gas equation $pV = nRT$, it follows that the total amount of substance in both vessels is the same before connection, that is, $n_1 = n_2 = n$.

After connection, the gas mixtures tend to reach equilibrium. This state can be characterized by equal partial pressures of individual gases in both vessels. At the same time, the total pressure is still given by the weights of the pistons and remains constant; there is no bulk gas flow, only spontaneous diffusion equalizing the partial pressures.

Denote quantities describing the state in the vessels after connection by an asterisk $*$. Consider, for example, vessel 1. During diffusion at constant total pressure and temperature, equimolar exchange of molecules occurs. This means that the amount of substance of nitrogen leaving vessel 1 is exactly equal to the amount of substance of oxygen entering it. It follows directly that the total change in the amount of substance in vessel 1 is zero, and the same holds for the second vessel.

Thus, for the new partial pressures, we can write

$$p^{*1}_{N_2} = p^{*2}_{N_2}\,,\quad p^{*1}_{O_2} = p^{*2}_{O_2}\,.$$

At the same time, the total pressure is the same everywhere and

$$p^{*1}_{N_2} + p^{*1}_{O_2} = p^{*2}_{N_2} + p^{*2}_{O_2} = p = 160\,\mathrm{Pa}\,.$$

The only solution to this system, due to the conservation of the amount of substance of the gases, is

$$p^{*1}_{N_2} = p^{*2}_{N_2} = p^{*1}_{O_2} = p^{*2}_{O_2} = 80\,\mathrm{Pa}\,.$$

Thus, the total changes of partial pressures in the individual vessels cancel each other.

Therefore, the total amount of substance in the vessels is conserved, $n_1 = n_1^* = n_2 = n_2^*$. At the same time, the temperature $T^* = T$ and the pressure $p^* = p$ are conserved, and we obtain

$$V_1^* = \frac{n_1^* R T^*}{p^*} = \frac{n_1 R T}{p} = V_1\,, \quad V_2^* = \frac{n_2^* R T^*}{p^*} = \frac{n_2 R T}{p} = V_2\,,$$

that is, the pistons *do not move.*

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