Problem III.4 ... between the mirrors

7 points

Consider an axially symmetric magnetic field between two large "magnetic mirrors". The magnetic field lines connecting one mirror to the other become denser as they approach the edges. In all areas, the component of the magnetic field parallel to the axis of symmetry is much stronger than the perpendicular component, i.e., $B_{\parallel} \gg B_{\perp}$. At the midpoint between the mirrors, the magnetic induction reaches its minimum value B_{min}, while near the mirrors it reaches its maximum B_{max}.

For a charged particle moving in such a field, its magnetic moment $\mu = E_{\mathrm{kin,k}}/B$ is conserved, where $E_{\mathrm{kin,k}}$ is the kinetic energy associated with motion perpendicular to the symmetry axis. A particle is fired from the center at an angle θ with respect to the axis of symmetry. What is the condition for the particle to be reflected between the mirrors and remain trapped?

Suppose we fire a large number of particles from the center in random directions; that is, each direction on an imaginary sphere is equally probable. What fraction of these particles will remain confined between the mirrors?

Marek got lost in a mirror maze.

If a particle of mass m is emitted at an angle θ from the axis with speed v, it has a perpendicular component of kinetic energy $E_{0,k} = (1/2)mv^2 \sin^2\theta$ and a magnetic moment $\mu_0 = E_{0,k}/B_{\mathrm{min}}$. Because the magnetic field strength increases toward the edges and the magnetic moment is required to remain constant according to the problem statement, the perpendicular component of the kinetic energy must also increase. This occurs when the parallel kinetic energy is converted into perpendicular energy— the particle turns and moves more perpendicular to the field.

The law of energy conservation still holds; since the magnetic field always exerts a force perpendicular to the particle's motion, it does no work and only changes the direction of the velocity. The total kinetic energy, therefore, remains constant.

If the field at the edges is so strong that even redirecting the entire kinetic energy into the perpendicular direction is insufficient to satisfy the condition of magnetic-moment conservation, the particle must reverse its motion and is thus reflected. Therefore,

$$\frac{E_{\mathrm{kin}}}{B_{\mathrm{max}}} < \frac{E_{0,k}}{B_{\mathrm{min}}}\,,$$
$$\frac{B_{\mathrm{min}}}{B_{\mathrm{max}}} < \sin^2\theta\,.$$

By replacing the inequality with equality, we obtain the limiting angle θ_{m} below which the particle is still reflected.

For the second part of the problem, we launch particles in completely random directions with uniform coverage of the imaginary sphere at which we aim. From the inequality above, we find that for a particle to remain in the device, the emission angle relative to the axis must satisfy

$$\theta \in \left(\arcsin\sqrt{\frac{B_{\mathrm{min}}}{B_{\mathrm{max}}}}, \pi - \arcsin\sqrt{\frac{B_{\mathrm{min}}}{B_{\mathrm{max}}}}\right)\,.$$

From geometric reasoning, we determine that the probability of hitting the region corresponding to these angles equals the ratio of the spherical band bounded by these angles to the surface area of the entire sphere. We compute this as the ratio of the surface area of a unit sphere minus the areas of two spherical caps determined by the angle θ_m to the surface area of the entire unit sphere. This gives

$$\frac{4\pi(1)^2 - 2\cdot 2\pi(1)^2(1-\cos\theta_m)}{4\pi(1)^2} = \cos\theta_m = \sqrt{1-\frac{B_\mathrm{min}}{B_\mathrm{max}}}\,,$$

which is the desired ratio.

More formally, we can perform the same calculation by integration. Owing to symmetry, it suffices to carry out the computation over an appropriate quarter-sphere, and we obtain for the probability

$$p = \frac{\int_0^\pi \int_{\theta_m}^{\pi/2} \sin\theta \,\mathrm{d}\theta \,\mathrm{d}\varphi}{\int_0^\pi \int_0^{\pi/2} \sin\theta \,\mathrm{d}\theta \,\mathrm{d}\varphi} = \frac{\pi\cos\theta_m}{\pi} = \sqrt{1-\frac{B_\mathrm{min}}{B_\mathrm{max}}}\,,$$

as expected.

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