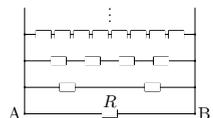## Problem III.2 . . . powerful resistance 3 points

We have a circuit with resistors arranged as shown in the diagram. The parallel configuration consists of infinitely many branches. In the N-th branch, there are 2^N resistors connected in series, each with a resistance of $R = 1\,\Omega$. What is the total resistance between points A and B?

David tried to think of a problem involving infinite resistance circuits.

First, let us notice how the resistors in the circuit are arranged. Each "level" forms a series connection of resistors; these levels are then connected to each other in parallel.

On the first conductor between points A and B, the resistance is R according to the problem statement. On the next conductor, the resistors are connected in series, so their equivalent resistance is $2R$. Each subsequent floor always has twice the resistance of the previous one. The resistances of the individual floors thus form the sequence

$$\{a_n\}_{n=1}^{\infty} = (R, 2R, 4R, 8R, \dots)\,.$$

Because the individual levels are connected in parallel, to determine the total resistance R_c we do not sum the level resistances directly; instead, we sum their reciprocals (conductances) as

$$\frac{1}{R_c} = \sum_i^{\infty} \frac{1}{R_i} = \frac{1}{R} + \frac{1}{2R} + \frac{1}{4R} + \dots$$

Factoring out $1/R$, we obtain the sum

$$\frac{1}{R_c} = \frac{1}{R}\left(1 + \frac{1}{2} + \frac{1}{4} + \dots\right).$$

Here, inside the parentheses, we recognize the well-known geometric series $\sum_{n=1}^{\infty} 1/2^n$, in which each successive term is half of the previous one. This type of series arises whenever a certain contribution repeatedly decreases by the same ratio—here always by one half, because the floor resistance doubles each time.

An infinite geometric series can be summed using the known formula

$$s = \frac{a_1}{1-q}\,,$$

where $a_1 = 1/R$ is the first term of the sequence and q is its ratio, that is, the number by which we obtain the next contribution in the sum (see the appendix for more details); in our case $q = 1/2$.

The result is therefore

$$\frac{1}{R_c} = \frac{1/R}{1 - 1/2} \quad \Rightarrow \quad R_c = \frac{1}{2}R\,.$$

Appendix on Sequences and Series

In the solution itself, the series appeared rather abruptly; we therefore provide some intuition behind the series.

A sequence is a collection of objects (for us, mainly real numbers) in which the order matters, and elements may repeat. We can define it either by listing the elements or by a rule—a geometric sequence is a specific example of the second type, where the first term is

given, and each subsequent term is obtained by multiplying the immediately preceding term by the *ratio*, a characteristic number that is the same for the entire sequence. For a geometric sequence, the nth term a_n is given by

$$a_n = a_1 q^{n-1},$$

where a_1 is the first term and q is the ratio.

A series is simply the sum of a sequence. For a finite geometric series with n terms, we now derive how to compute its sum. The sum s_n has the general form

$$s_n = a_1 + a_2 + \cdots + a_n,$$

which can be rewritten as

$$s_n = a_1 + a_1 q + a_1 q^2 + \cdots + a_1 q^{n-1}.$$

We can multiply the entire equation by $-q$, obtaining

$$-s_n q = -a_1 q - a_1 q^2 - a_1 q^3 - \cdots - a_1 q^n.$$

Adding the two previous equations gives

$$s_n - s_n q = a_1 - a_1 q^n$$
$$s_n(1-q) = a_1(1 - q^n)$$
$$s_n = a_1 \frac{1 - q^n}{1 - q}.$$

The final step is to pass from the finite series to the infinite one. First, consider when it makes sense to speak about the sum, depending on the value of the ratio q: if $|q| > 1$, each successive term of the sequence is larger than the previous one, and therefore the total sum "escapes" to infinity (formally, we say that the series *diverges*). An infinite series has a finite sum only if the contribution of each successive term decreases, that is, if $|q| < 1$.

Let us examine the expression

$$s_n = a_1 \frac{1 - q^n}{1 - q}$$

again and consider its behavior as n approaches infinity[1]. We consider only series that converge, so we know that $|q| < 1$. The only part depending on n is q^n. Because $|q| < 1$, this quantity decreases as n increases, until it "cannot be distinguished from zero". Therefore, the sum of the infinite geometric series is

$$s_\infty = a_1 \frac{1 - 0}{1 - q} = \frac{a_1}{1 - q}.$$

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[1]Formally, we would compute the limit of this expression. For now, intuitive reasoning is sufficient.