Problem III.1 ... (un)stable toilet paper

3 points

Danka is interested in how the stability of a toilet-paper roll changes as part of the paper is used. She therefore took a new roll containing $N = 120$ sheets and measured its parameters. She found that the roll has height $h = 9.5\,\mathrm{cm}$, diameter $d_1 = 12.5\,\mathrm{cm}$, and mass $m_1 = 139\,\mathrm{g}$. The cardboard tube on which the paper is wound has a diameter $d_0 = 3.5\,\mathrm{cm}$ and mass $m_0 = 4\,\mathrm{g}$. By what factor does the potential energy required to tip the roll decrease when only one-third of the paper remains compared to the initial state? Assume the layers of paper are wound uniformly and tightly, without volumetric deformation.

Danka forgot to take her phone when she went to relieve herself.

The stability of a mechanical system (in our case, the paper roll itself) is the difference in potential energy between the unstable and the stable equilibrium positions. It therefore indicates how much energy (at a minimum) must be supplied to the roll to tip it over. For the roll, the unstable position is the state in which it is tilted so that its center of mass (which, due to symmetry, is at the center of the roll) lies above the pivot point. Let us first calculate the stability of the original roll. Its potential energy when standing on the surface in the stable position is

$$E_{01} = m_1 g \frac{h}{2}\,.$$

Let us denote the distance of the center of mass from the pivot point by a_1. Then, when tilted into the unstable position, its potential energy is

$$E_1 = m_1 g a_1 = m_1 g \sqrt{\left(\frac{h}{2}\right)^2 + \left(\frac{d_1}{2}\right)^2}\,.$$

The stability of the original roll is therefore

$$S_1 = E_1 - E_{01}\,,$$
$$S_1 = \frac{m_1 g}{2}\left(\sqrt{h^2 + d_1^2} - h\right)\,,$$
$$S_1 \doteq 42.27\,\mathrm{mJ}\,.$$

To calculate the stability of the roll with one-third of the original amount of paper, let us examine the change in its mass and dimensions. The original paper mass is $m_{p1} = m_1 - m_0$. One third of this remains on the roll, so the resulting roll mass is the sum of this mass and the mass of the cardboard tube,

$$m_2 = \frac{m_1 - m_0}{3} + m_0 = \frac{m_1 + 2m_0}{3}\,.$$

The potential energy of the roll with one-third of the paper in the stable position is then

$$E_{02} = m_2 g \frac{h}{2}\,.$$

To calculate the potential energy in the unstable position, we need the distance between the roll's center of mass and the pivot point,

$$a_2 = \sqrt{\left(\frac{h}{2}\right)^2 + \left(\frac{d_2}{2}\right)^2}\,.$$

For this, we must determine the diameter of the roll with one-third of the paper, d_2. The key point is that removing two-thirds of the paper length simultaneously removes two-thirds of its volume. The original paper volume is

$$V_{p1} = \pi h \left[\left(\frac{d_1}{2}\right)^2 - \left(\frac{d_0}{2}\right)^2 \right] .$$

The volume of one-third of the original paper is then

$$V_{p2} = \pi h \left[\left(\frac{d_2}{2}\right)^2 - \left(\frac{d_0}{2}\right)^2 \right] = \frac{1}{3} V_{p1} .$$

Substituting the expression for V_{p1} and simplifying, we obtain

$$\begin{aligned} \frac{\pi h}{4}(d_2^2 - d_0^2) &= \frac{1}{3}\frac{\pi h}{4}(d_1^2 - d_0^2) , \\ d_2^2 &= \frac{d_1^2 - d_0^2}{3} + d_0^2 , \\ d_2 &= \sqrt{\frac{d_1^2 + 2d_0^2}{3}} , \\ d_2 &\doteq 7.76\,\mathrm{cm} . \end{aligned}$$

Thus, the potential energy of the roll with one-third of the paper in the unstable position is

$$\begin{aligned} E_2 &= m_2 g a_2 , \\ E_2 &= \frac{m_1 + 2m_0}{3} \frac{g}{2} \sqrt{h^2 + \left(\frac{d_1^2 + 2d_0^2}{3}\right)} . \end{aligned}$$

The stability of the roll with one-third of the paper is then

$$\begin{aligned} S_2 &= E_2 - E_{02} , \\ S_2 &= \frac{m_1 + 2m_0}{3} \frac{g}{2} \left(\sqrt{h^2 + \frac{d_1^2 + 2d_0^2}{3}} - h \right) , \\ S_2 &\doteq 6.65\,\mathrm{mJ} . \end{aligned}$$

Finally, the ratio of the quantities S_1 and S_2 is

$$\frac{S_1}{S_2} \doteq 6.4 .$$

Thus, tipping the roll with one-third of the original amount of toilet paper requires 6.4 times less energy.

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FYKOS is organized by students of Faculty of Mathematics and Physics of Charles University. It's part of Media Communications and PR Office and is supported by Institute of Theoretical Physics of CUNI MFF, his employees and The Union of Czech Mathematicians and Physicists. The realization of this project was supported by Ministry of Education, Youth and Sports of the Czech Republic.