Problem II.5 ... induced interaction

10 points

Consider an immovable, homogeneous, uncharged, conducting sphere of radius R. A particle carrying charge q is launched from infinity with speed v toward the sphere, along a trajectory characterized by an impact parameter b (i.e., the perpendicular distance from the sphere's center to the asymptotic path of the particle).

Determine the conditions under which the particle does not collide with the uncharged sphere. Neglect the effect of any induced magnetic field.

Jarda wanted to simulate molecule interactions.

To solve the problem, we use the method of image charges for a conducting sphere[1] and the method of the effective potential for motion in a spherically symmetric potential[2]. This allows us to separate the motion in the radial and angular directions, thereby effectively turning the problem into a one-dimensional one. We can then view the problem as the motion of a particle in a one-dimensional potential, which is easy to visualize.

The field induced by the conducting sphere can be replaced by the field of an image charge q' located somewhere inside the sphere and an image charge q'' located at the center of the sphere. The position of the charge q' is the image of the position of the real charge under spherical inversion. Let $\mathbf{r}$ and $\mathbf{r}'$ denote, respectively, the position vectors of the charges q and q' with respect to the origin placed at the center of the conducting sphere. Then

$$rr' = R^2 \quad \Rightarrow \quad r' = \frac{R^2}{r} \,.$$

First, let us assume that the sphere is grounded and the charge at the center is zero. The total electric potential is to be zero on the sphere's surface. In particular, it is therefore zero at the intersection of the line connecting the charges with the surface of the conducting sphere, where we obtain the condition

$$\frac{1}{4\pi\varepsilon_0}\left(\frac{q}{r-R} + \frac{q'}{R-r'}\right) = 0 \,,$$

$$q' = -\frac{R-r'}{r-R}q = -\frac{R}{r}q \,.$$

The real charge thus moves as if it were in the potential created by the image charge

$$V(\mathbf{r}) = q\varphi(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\frac{qq'}{r-r'} = -\frac{q^2}{4\pi\varepsilon_0}\frac{R}{r^2-R^2} \,.$$

In this scenario, however, to an observer outside the sphere, the sphere appears to carry charge q'. To eliminate this total charge, we may place a charge $q'' = -q'$ at the center of the sphere, which creates an additional Coulomb field. This certainly does not disturb the constancy of the potential on the sphere (which is precisely the condition for the sphere to be conducting), and it therefore clearly solves our original problem. The total potential is thus of the form

$$V(\mathbf{r}) = -\frac{q^2}{4\pi\varepsilon_0}\frac{R}{r^2-R^2} + \frac{q^2}{4\pi\varepsilon_0}\frac{R}{r^2} \,.$$

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[1] `https://en.wikipedia.org/wiki/Method_of_image_charges`

[2] `https://en.wikipedia.org/wiki/Effective_potential`

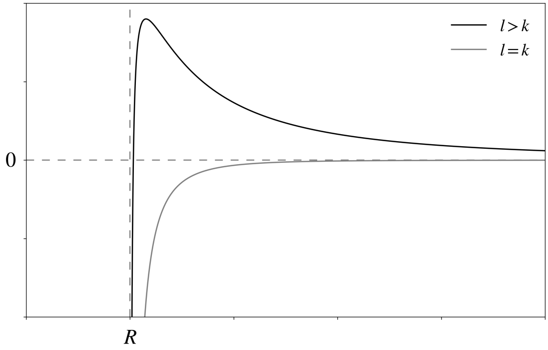

Figure 1: The effective potential V_{eff} (vertical axis) as a function of the position r (horizontal axis) for $l > k$ (the curve with a maximum) and $l = k$ (the monotonically increasing curve).

The potential depends only on the charge position, and the situation is spherically symmetric. We may therefore introduce the effective potential

$$V_{\mathrm{eff}}(\mathbf{r}) = V(\mathbf{r}) + \frac{L^2}{2mr^2} = -\frac{q^2}{4\pi\varepsilon_0}\frac{R}{r^2 - R^2} + \frac{q^2}{4\pi\varepsilon_0}\frac{R}{r^2} + \frac{L^2}{2mr^2} = -\frac{k^2}{r^2 - R^2} + \frac{l^2}{r^2}\,, \qquad (1)$$

where we have introduced the constants $k = \sqrt{(q^2R)/(4\pi\varepsilon_0)}$ and $l = \sqrt{k^2 + L^2/(2m)}$.

The solution of the problem now consists in discussing the behavior of the effective potential $V_{\mathrm{eff}}(\mathbf{r})$, which is shown in Figure 1. Clearly, $V_{\mathrm{eff}} \to -\infty$ as $r \to R^+$ and $V_{\mathrm{eff}} \to 0$ as $r \to \infty$. We are interested only in the behavior on the interval (R, ∞), where there are no further divergences. For the following considerations, let us compute the first derivative of the potential:

$$\frac{\mathrm{d}V_{\mathrm{eff}}}{\mathrm{d}r} = \frac{2k^2r}{(r^2 - R^2)^2} - \frac{2l^2}{r^3}\,.$$

To find the stationary points, we set the derivative equal to zero:

$$\frac{2k^2r}{(r^2 - R^2)^2} - \frac{2l^2}{r^3} = 0 \quad \Rightarrow \quad (l^2 - k^2)r^4 - 2R^2l^2r^2 + R^4l^2 = 0\,, \qquad (2)$$

which is (as long as $k \neq l$) a quadratic equation in r^2. For r we obtain the roots

$$r = R\sqrt{\frac{l}{l \pm k}}\,,$$

but we seek only roots satisfying $r > R$, from which we immediately infer that only the root $r_0 = R\sqrt{l/(l - k)}$ satisfies this condition, and the condition $l > k$ must hold. However, by the definition of l, this is always satisfied as long as $k \neq l$.

The discussion, therefore, splits into two cases.

i) For $l > k$, there exists a local maximum inside the interval (R, ∞). There is therefore a critical energy $E_0 = E(r_0)$. For $E \leq E_0$, the particle, in terms of the radial component of motion, stops and reverses the direction of its motion before reaching this maximum (or remains at it), and it cannot collide with the sphere. For $E > E_0$, the maximum energy can be overcome, and the particle collides with the sphere.

ii) For $l = k$, from (2) we obtain

$$-2R^2l^2r^2 + R^4l^2 = 0 \quad \Rightarrow \quad r = \frac{R}{\sqrt{2}}\,.$$

The extremum thus does not lie in the interval (R, ∞). The potential is increasing and negative over the entire interval (R, ∞). For $E < 0$, the particle clearly cannot exist at infinity, and this case is irrelevant for our problem. For $E \geq 0$, the value of the effective potential decreases to minus infinity, and the particle therefore collides with the sphere.

Altogether, we have found that the particle does not collide with the sphere precisely when $l > k$ (which corresponds to $L \neq 0$) and simultaneously $E \leq E_0$.

By substituting r_0 into equation (1), we express the critical energy:

$$E_0 = V_{\text{eff}}(r_0) = -\frac{k(l-k)}{R^2} + \frac{l(l-k)}{R^2} = \frac{(l-k)^2}{R^2}\,.$$

Now consider a particle at position $\mathbf{r}$ with velocity $-v\mathbf{e}_x$ at a vertical distance b from the center of the sphere. Such a particle has angular momentum

$$\mathbf{L} = \mathbf{r} \times \mathbf{p} = -mv(\mathbf{r} \times \mathbf{e}_x) \quad \Rightarrow \quad L = mvr\sin\theta\,,$$

Clearly, $\sin\theta = b/r$, and therefore

$$L = mvr\,\frac{b}{r} = mvb\,. \tag{3}$$

The angular momentum of the particle is conserved in a spherically symmetric problem. From equation (3), applied at launch from infinity, we obtain $L = mvb$, where v and b are the initial speed and impact parameter. For the parameter l, we therefore obtain

$$l = \sqrt{k^2 + \frac{L^2}{2m}} = \sqrt{k^2 + \frac{1}{2}mv^2b^2}\,.$$

The total energy of the particle is given by its kinetic energy at infinity. For the particle not to collide with the sphere, we must have $L \neq 0$ and $T \leq E_0$, where $T = (1/2)mv^2$ is the initial kinetic energy. We satisfy the first condition if $v \neq 0$ and simultaneously $b \neq 0$. The second condition becomes

$$\frac{1}{2}mv^2 \leq \frac{1}{R^2}\left(\sqrt{k^2 + \frac{L^2}{2m}} - k\right)^2 \quad \Longleftrightarrow \quad \frac{1}{2}mv^2(b^2 - R^2) \geq 2k\sqrt{k^2 + \frac{1}{2}mv^2b^2} - 2k^2\,, \tag{4}$$

where

$$2k\sqrt{k^2 + \frac{1}{2}mv^2b^2} - 2k^2 = 2k\left(\sqrt{k^2 + \frac{1}{2}mv^2b^2} - k\right) \geq 0\,. \tag{5}$$

From equations (5) and (4), it is evident that the inequality can be satisfied only if $|b| \geq R$. In that case, we may square the inequality, thereby obtaining a quadratic inequality for $T = (1/2)mv^2$ in the form

$$T(b^2 - R^2) + 2k^2 \geq 2k\sqrt{k^2 + Tb^2} \quad \Rightarrow \quad (b^2 - R^2)^2 T^2 - 4k^2 b^2 T + 4k^2(b^2 - R^2)T \geq 0\,,$$

which we may immediately divide by T (which is positive because $v \neq 0$), thereby obtaining

$$(b^2 - R^2)T - 4k^2R^2 \geq 0 \quad \Rightarrow \quad (b^2 - R^2)T \geq 4k^2R^2\,. \tag{6}$$

The inequality (6) clearly cannot be satisfied if $|b| < R$. For $|b| > R$, we obtain

$$T \geq \frac{4k^2R^2}{b^2 - R^2} \iff v^2 \geq \frac{1}{4\pi\varepsilon_0}\frac{8q^2}{m}\frac{R^3}{b^2 - R^2} \tag{7}$$

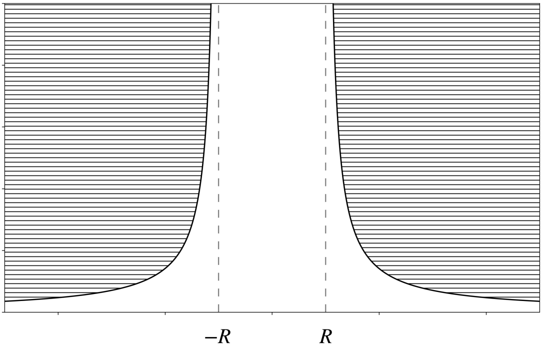

Figure 2: The parameter space of b (horizontal axis) and v (vertical axis) for $v \geq 0$. The shaded part of the graph corresponds to parameter settings for which the particle does not collide with the sphere. This set also includes its boundary in the form of two black curves.

It remains to resolve the case $|b| = R$. Here condition (6) states that $k^2 \leq 0$, that is, $k = 0$, which can occur only for zero charge q or zero radius R. Altogether, we have thus found that the particle does not collide with the sphere if $|b| > R$ and simultaneously the inequality (7) is satisfied, as shown in Figure 2.

Finally, a few remarks. In the case $E = E_0$ for $l > k$, the particle ends up in a circular orbit, which is, however, unstable. An arbitrarily small radial impulse would therefore suffice to deflect it either back to infinity or into the clutches of the charged sphere. If we do not consider any such impulse (which is unrealistic, but consistent with the problem statement), then the particle indeed remains in the circular orbit, and it does not collide with the sphere.

Let us also note that at high speeds, it is necessary to take into account both relativistic effects and the effect of the induced magnetic field, which makes the problem highly nontrivial.

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