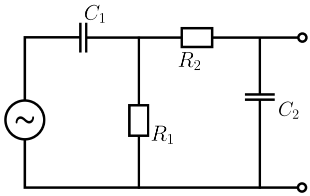
Problem II.4 ... frequency filter 8 points

Consider a circuit as shown in the diagram. Determine the range of source frequencies of the alternating voltage for which the RMS voltage across the capacitor C_2 is at least half of its maximum attainable value.



Jarda's favourite RLC circuits have returned after a while...

As part of the solution, we first express in general the voltage across the capacitor U_{C_2} as a function of the system parameters and the angular frequency ω. We then find its maximum value and finally determine all frequencies at which the voltage is at least one half of this maximum value.

The problem asks for the effective voltage. For a harmonic alternating signal, the effective voltage U_{ef} is related to the amplitude U_{am} by $\sqrt{2}\,U_{\mathrm{ef}} = U_{\mathrm{am}}$. This corresponds to such a direct voltage U_{ef} that would deliver the same power to a resistor as an alternating voltage with amplitude U_{am}. Since the relation between effective and maximum voltage is linear, we do not need to distinguish between them in solving this problem: we find the maximum and then determine the frequency interval where the voltage is at least one half of this maximum. It therefore makes no difference whether we work with effective or amplitude values, and we will not distinguish between these quantities further. We adopt analogous notation for currents.

Even when solving circuits with alternating voltage, we may use Kirchhoff's laws. In each loop of the circuit, the total sum of voltages across individual elements must be zero, and similarly, at each node where several conductors meet, the current flowing in must equal the current flowing out. Let us denote the amplitude of the current through the voltage source by I, the amplitude through resistor R_1 by I_1, and analogously the amplitude through resistor R_2 by I_2. Then

$$I = I_1 + I_2\,.$$

At the same time, the voltage across resistor R_1 satisfies

$$R_1 I_1 = U - \frac{-j}{C_1\omega} I\,.$$

Here we used the fact that the voltage across a resistor R carrying a current I is, according to Ohm's law, $U = RI$. A similar relation also holds for a capacitor: the instantaneous voltage is proportional to the instantaneous current but phase-shifted, which we express using the imaginary unit j (instead of the mathematical symbol i, to avoid confusion with the current symbol),

$$U = \frac{-j}{C\omega} I\,.$$

For further information and a derivation of these relations, we refer the reader to other textbooks.

This voltage is simultaneously present in the second loop, where

$$R_1 I_1 = R_2 I_2 + \frac{-j}{C_2\omega} I_2 \,.$$

We have obtained three equations for the three unknowns I, I_1, and I_2. Substituting the current I from the first Kirchhoff law into the voltage equation of the first loop yields

$$\left(R_1 - \frac{j}{C_1\omega}\right) I_1 = U + \frac{j}{C_1\omega} I_2 \,,$$

from which we can substitute for I_1 using the last equation,

$$\left(R_1 - \frac{j}{C_1\omega}\right)\left(R_2 - \frac{j}{C_2\omega}\right) I_2 = U R_1 + \frac{jR_1}{C_1\omega} I_2 \,,$$

$$\left(\left(R_1 - \frac{j}{C_1\omega}\right)\left(R_2 - \frac{j}{C_2\omega}\right) - \frac{jR_1}{C_1\omega}\right) I_2 = U R_1 \,.$$

The voltage across capacitor C_2 is then

$$U_{C_2} = -\frac{j}{C_2\omega} I_2 = \frac{-jU}{\left(1 - \frac{j}{C_1 R_1 \omega}\right)\left(R_2 C_2 \omega - j\right) - \frac{j}{C_1} C_2} =$$

$$= \frac{-jU}{R_2 C_2 \omega - \frac{1}{C_1 R_1 \omega} - j\left(1 + \frac{R_2 C_2}{C_1 R_1} + \frac{C_2}{C_1}\right)} \,.$$

We see that, for simplicity of notation, we can introduce the time constants of the individual loops as $\tau_1 = R_1 C_1$ and $\tau_2 = R_2 C_2$. It is also evident that the relation between the voltage U and the sought voltage across capacitor C_2 is linear, and that the ratio of their magnitudes is given by the absolute value of the expression

$$\frac{|U|}{|U_{C_2}|} = \left|\tau_2\omega - \frac{1}{\tau_1\omega} - j\left(1 + \frac{\tau_2}{\tau_1} + \frac{C_2}{C_1}\right)\right| = \sqrt{\left(\tau_2\omega - \frac{1}{\tau_1\omega}\right)^2 + \left(1 + \frac{\tau_2}{\tau_1} + \frac{C_2}{C_1}\right)^2} \,.$$

We observe that as $\omega \to \infty$ this ratio grows without bound, and similarly for $\omega \to 0$. Only for intermediate frequencies is the voltage across capacitor C_2 not very small. This is therefore a circuit in which the voltage depends on frequency, and by choosing the parameters of the individual components we can influence the tuning of the circuit.

It is clear that the maximum voltage across the capacitor occurs at the frequency for which the value under the square root is minimal. We cannot affect the constant term, so we must minimize the expression $(\tau_2\omega - 1/\tau_1\omega)^2$. This expression can be zero, which occurs at the frequency

$$\omega_{\max} = (\tau_1\tau_2)^{-1/2} \,.$$

The maximum voltage across capacitor C_2 is therefore

$$|U_{\max}| = \frac{1}{\sqrt{\left(1 + \frac{\tau_2}{\tau_1} + \frac{C_2}{C_1}\right)^2}} |U| = \frac{1}{1 + \frac{\tau_2}{\tau_1} + \frac{C_2}{C_1}} |U|$$

According to the problem statement, the voltage across capacitor C_2 must be at least one half of this value. We therefore compare

$$\frac{1}{2}\frac{1}{\sqrt{\left(1+\frac{\tau_2}{\tau_1}+\frac{C_2}{C_1}\right)^2}} < \frac{1}{\sqrt{\left(\tau_2\omega-\frac{1}{\tau_1\omega}\right)^2+\left(1+\frac{\tau_2}{\tau_1}+\frac{C_2}{C_1}\right)^2}}\,,$$

$$\sqrt{\left(\tau_2\omega-\frac{1}{\tau_1\omega}\right)^2+\left(1+\frac{\tau_2}{\tau_1}+\frac{C_2}{C_1}\right)^2} < 2\sqrt{\left(1+\frac{\tau_2}{\tau_1}+\frac{C_2}{C_1}\right)^2}\,,$$

$$\left(\tau_2\omega-\frac{1}{\tau_1\omega}\right)^2 < 3\left(1+\frac{\tau_2}{\tau_1}+\frac{C_2}{C_1}\right)^2\,.$$

We have thus obtained an inequality of the form $x^2 < y^2$, which can be rewritten as $|x| < |y|$. If we additionally know that $y > 0$, we can further reduce this to $|x| < y$, which means $-y < x < y$.

Since the expression $\sqrt{3}(1+\tau_2/\tau_1+C_2/C_1)$ is positive, we can perform the same steps here and obtain

$$-\sqrt{3}\left(1+\frac{\tau_2}{\tau_1}+\frac{C_2}{C_1}\right)\omega < \tau_2\omega^2-\frac{1}{\tau_1} < \sqrt{3}\left(1+\frac{\tau_2}{\tau_1}+\frac{C_2}{C_1}\right)\omega\,.$$

These represent two quadratic inequalities in ω that must hold simultaneously. We solve them separately and, for simplicity, introduce the notation

$$A = \frac{1}{2\tau_2}\sqrt{3\left(1+\frac{\tau_2}{\tau_1}+\frac{C_2}{C_1}\right)^2+4\frac{\tau_2}{\tau_1}} \qquad \text{and} \qquad B = \frac{1}{2\tau_2}\sqrt{3}\left(1+\frac{\tau_2}{\tau_1}+\frac{C_2}{C_1}\right)\,.$$

Solving the first inequality

$$-\sqrt{3}\left(1+\frac{\tau_2}{\tau_1}+\frac{C_2}{C_1}\right)\omega < \tau_2\omega^2-\frac{1}{\tau_1}$$

yields

$$\omega > (A-B) \qquad \text{or} \qquad \omega < (-A-B)\,,$$

where the second interval is excluded because we are interested only in positive ω.

Solving the second inequality

$$\tau_2\omega^2-\frac{1}{\tau_1} < \sqrt{3}\left(1+\frac{\tau_2}{\tau_1}+\frac{C_2}{C_1}\right)\omega$$

gives

$$(-A+B) < \omega < (A+B)\,.$$

Finally, noting that $A > B$, we deduce that the values of ω satisfying both inequalities (and thus the condition of the problem) lie in the interval

$$(A-B) < \omega < (A+B)\,,$$

that is,

$$\omega > \frac{1}{2\tau_2}\left(\sqrt{3\left(1+\frac{\tau_2}{\tau_1}+\frac{C_2}{C_1}\right)^2+4\frac{\tau_2}{\tau_1}}-\sqrt{3}\left(1+\frac{\tau_2}{\tau_1}+\frac{C_2}{C_1}\right)\right)$$

and simultaneously

$$\omega < \frac{1}{2\tau_2}\left(\sqrt{3\left(1+\frac{\tau_2}{\tau_1}+\frac{C_2}{C_1}\right)^2+4\frac{\tau_2}{\tau_1}}+\sqrt{3}\left(1+\frac{\tau_2}{\tau_1}+\frac{C_2}{C_1}\right)\right).$$

There thus indeed exists a frequency interval in which the voltage across capacitor C_2 is not significantly lower than the source voltage. At these frequencies, the circuit acts as a voltage-pass filter. Outside this interval, the attenuation is effective and those frequencies are filtered out. In this way, we can construct a circuit that passes only selected frequencies, which can be useful in a variety of electronic applications.

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FYKOS is organized by students of Faculty of Mathematics and Physics of Charles University. It's part of Media Communications and PR Office and is supported by Institute of Theoretical Physics of CUNI MFF, his employees and The Union of Czech Mathematicians and Physicists. The realization of this project was supported by Ministry of Education, Youth and Sports of the Czech Republic.