Problem II.1 ... do not lose any time — 3 points

Jarda is 110 m *away from a traffic light showing red and is approaching it at a speed of* 50 km/h. *He intends to decelerate with an acceleration of no more than* $3.0\,\mathrm{m{\cdot}s^{-2}}$.

What is the maximum speed at which he can pass the traffic light while it is green, assuming he never presses the accelerator during the entire approach? Assume that when Jarda is not actively braking, the car moves at a constant speed. The green light will turn on in 21 s.

Jarda was in a big hurry.

What must the deceleration profile be for Jarda to pass with the highest possible speed? Consider a graph of velocity versus time. At the moment the green light flashes, $\tau = 21\,\mathrm{s}$, Jarda's speed should be maximal. At $t = 0$, the initial speed is $v_0 = 50\,\mathrm{km{\cdot}h^{-1}}$, and the area under the velocity curve must equal the distance $s = 110\,\mathrm{m}$.

If Jarda passes the intersection at speed v_1, the area of the rectangle under the constant function $v = v_1$ is $v_1\tau$. To maximize v_1, the area outside this rectangle must be minimized. This area corresponds to deceleration from v_0 to v_1. The greater the acceleration magnitude, the sooner the car reaches v_1, and the smaller the area outside the rectangle. Therefore, Jarda should decelerate as quickly as possible to v_1 and then maintain this speed.

The deceleration time for acceleration a is

$$\tau_1 = \frac{v_0 - v_1}{a}\,,$$

and the total distance traveled, from geometric considerations, is

$$s = v_1\tau + \frac{1}{2}\tau_1\left(v_0 - v_1\right)\,.$$

From these two equations, we can solve for τ_1 and v_1. Substituting τ_1 from the first equation into the second yields a quadratic equation

$$2s\,a = 2v_1\tau a + \left(v_0 - v_1\right)^2\,.$$

Expanding and rearranging yields

$$v_1^2 + 2\left(\tau a - v_0\right)v_1 + v_0^2 - 2s\,a = 0\,,$$

so that

$$v_1 = v_0 - a\,\tau \pm \sqrt{\left(\tau a - v_0\right)^2 - \left(v_0^2 - 2s\,a\right)}\,.$$

To maximize v_1, we take the solution with the plus sign:

$$v_1 = v_0 - a\,\tau + \sqrt{\left(v_0 - \tau a\right)^2 - \left(v_0^2 - 2s\,a\right)} \doteq 16.4\,\mathrm{km{\cdot}h^{-1}}\,.$$

This solution is valid only if Jarda needs to decelerate, i.e.,

$$v_0\tau > s\,,$$

which is satisfied here. Otherwise, he would pass at the initial speed v_0.

It is also possible that Jarda might fail to decelerate in time and run a red light. This occurs if

$$v_0\tau - \frac{1}{2}a\,\tau^2 > s\,,$$

which, in this case, does not happen.

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