Problem I.5 ... eliptical lens   9 points

Consider a stretched rotary ellipsoid from a seethrough material with a refraction index $n = 1.5$. Consider a plane that is parallel to the rotary axis and intersects the ellipsoid near the adjacent vertex. We cut through this plane and make the analogous cut at the other side. We stick both of the cut pieces together at the flat sides such that we get a narrow lens. How does a point at the distance of $d = 4\,\mathrm{m}$ that lies at the optical axis display from the center of the lens? Projection of the original body is an ellipse on the given scheme with semi-major axis $a = 3\,\mathrm{m}$ and semi-minor axis $b = 2\,\mathrm{m}$.   *Jarda was inspired by nurofen medicine.*

In a given region, an ellipsoid has different radii of curvature in different cross sections. This is a substantial difference compared to the case of a spherical interface, where spherical symmetry can be exploited and the problem can be treated in a single plane. Here, by contrast, we must analyze the imaging produced by each cross section of the lens separately.

We first express all radii of curvature of a single lens–air interface as functions of a parameter. We introduce a Cartesian coordinate system and describe the surface of the ellipsoid as

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{b^2} = 1\,,$$

where the ellipsoid is rotationally symmetric about the x axis, along which it has its greatest extent. We now intersect the ellipsoid with the plane

$$x \sin\varphi - y \cos\varphi = 0\,,$$

which passes through the origin and is perpendicular to the plane $z = 0$. The parameter describing its rotation about the z axis is the angle φ, defined as the angle between its line of intersection with the plane $z = 0$ and the x axis. The intersection of this plane with the ellipsoid is an ellipse. Its co-vertex is simultaneously the vertex of our thin lens.

The length of the semi-major axis of this ellipse is

$$c = \sqrt{\frac{a^2 b^2}{b^2 + a^2 \tan^2\varphi} + \frac{a^2 b^2 \tan^2\varphi}{b^2 + a^2 \tan^2\varphi}} = \frac{ab}{\sqrt{b^2 \cos^2\varphi + a^2 \sin^2\varphi}}\,,$$

which follows from the Pythagorean theorem. It is the distance of the point where the vertical plane intersects the ellipsoid in the plane $z = 0$ from the origin.

The length of the semiaxis c therefore depends on the angle φ, that is, on the direction in which the ellipsoid is cut. Since we are dealing with a thin lens and work within the approximation of geometrical optics, we approximate the ellipse at its co-vertex by its osculating circle. Its radius is

$$R = \frac{c^2}{b} = \frac{a^2 b}{b^2 \cos^2\varphi + a^2 \sin^2\varphi}\,.$$

The lens thus possesses a continuous set of radii of curvature. Within a given cross section of the lens, the focal length can be determined from the general relation for a thin lens,

$$f = \frac{n_1 R_1 R_2}{(n_2 - n_1)(R_2 - R_1)}\,,$$

where n_1 is the refractive index of the surrounding medium, n_2 is the refractive index of the lens, and R_1 and R_2 are the radii of curvature of the first and second interfaces. In our case we

may take $n_1 = 1$ for a lens in air, $n_2 = n$, and $R_2 = -R_1 = -R$, since both halves of the lens are identical and we adopt the sign convention in which the object distance in front of the lens is positive, the image distance behind the lens is positive, the radius of curvature is positive if the interface is convex toward the object, and the focal length of a converging lens is positive. The focal length of the lens as a function of the cutting direction is then

$$f = \frac{R}{2(n-1)} = \frac{a^2 b}{2(n-1)(b^2 \cos^2 \varphi + a^2 \sin^2 \varphi)} \,. \tag{1}$$

The lens therefore exhibits a continuous range of focal lengths, from b up to a^2/b in expression (1).

To determine the image position of a given object we use the imaging equation in the form

$$d' = \frac{df}{d - f} \,,$$

where d is the object distance in front of the lens and d' is the image distance behind the lens. Substituting our focal lengths yields

$$d' = \frac{d}{\frac{2d(n-1)}{a^2 b}(b^2 \cos^2 \varphi + a^2 \sin^2 \varphi) - 1} \,.$$

We now analyze this result for the specific numerical values given in the problem statement, which after simplification becomes

$$d' = \frac{36}{18 \cos^2 \varphi + 8 \sin^2 \varphi - 9} \,\mathrm{m} = -\frac{36}{5 \cos 2\varphi - 4} \,\mathrm{m} \,.$$

This function has the range $(-\infty, -36\,\mathrm{m})$ and $\langle 4\,\mathrm{m}, \infty)$. For the given parameters, the source is therefore imaged onto two half-lines: one behind the lens, where a real image is formed, and one in front of the lens, where a virtual image is formed.

If the given distance d did not lie in the interval $\langle b, a^2/b \rangle$, the image of the source would be formed on a finite interval, either as a real or as a virtual image.

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