Problem I.4 ... cake problems

8 points

A cake on Jarda's plate sometimes topples over to its other side. What is the work needed for such misfortune? Consider polar symetric cake of weight M and of radius R, its center of mass is located in height h; solve universally for its cutting to n same pieces.
Bonus: *Bake your own cake and send us a photo and a recipe. The best will be shared in the problem solution!*

You are more likely to find Jarda in a patisserie than in school.

Let us assume that the cake has radius R, mass M, and that its center of mass is located at a height h. The mass of a single slice is then M/n, and the slice is a body whose base is a circular sector and whose height is perpendicular to this base. To compute the work required to tip a slice onto its side, it is necessary to determine by how much the center of mass must be raised for the slice to tip over spontaneously. Spontaneous tipping occurs at the moment when the center of mass of the slice lies vertically above the edge about which the slice rotates. If the center of mass is still slightly closer to the stable position, the slice will rotate back onto its base; if the critical point is exceeded, the slice will topple, because the torque due to gravity acting at the center of mass overcomes the reaction force of the plate acting at the contact edge.

From the above it is clear that the height of the cake itself is not required; only the position of its center of mass is relevant. For each slice, this center of mass is always at height h. However, the remaining coordinates of the center of mass must also be determined. By symmetry, the center of mass lies on the axis of symmetry of the slice. The remaining question is how far it is from the corner that coincided with the center of the entire cake before cutting. Let us denote the central angle by 2α, for which

$$2\alpha = \frac{2\pi}{n} \quad \Rightarrow \quad \alpha = \frac{\pi}{n}\,.$$

The radial coordinate of the center of mass of such a body can be found using formulas available on the internet; however, for completeness, we present the derivation it here. If we seek the position of the center of mass of an object that can be conceptually divided into i parts, then in a Cartesian coordinate system the position of the center of mass of the entire object is given by

$$\mathbf{r}_\mathrm{T} = \frac{\sum_i m_i \mathbf{r}_{\mathrm{T}i}}{\sum m_i} = \frac{\sum_i m_i \mathbf{r}_{\mathrm{T}i}}{m}\,,$$

where m is the total mass of the body under consideration.

We use this approach in our calculation. We conceptually divide the slice of cake into very thin circular sectors with vertex angle $\mathrm{d}\beta$. These shapes are sufficiently narrow that their bases can be approximated as isosceles triangles with equal sides of length R. The center of mass of such a triangle lies on its axis of symmetry at a distance $2/3$ of the length of the median measured from the vertex angle $\mathrm{d}\beta$. Because the triangle is very narrow, we may further assume that the length of this median is equal to R, so the center of mass of this triangle lies at a distance $2R/3$ from its vertex.

To determine the radial coordinate of the center of mass R_T, we reassemble the slice of cake from a very large number of these triangles. Since the slices are infinitesimally thin, we replace the sum by an integral of the form

$$R_\mathrm{T} = \frac{\int_{-\alpha}^{\alpha} \frac{2}{3} R \cos\beta \rho \frac{R^2}{2}\,\mathrm{d}\beta}{\int_{-\alpha}^{\alpha} \rho \frac{R^2}{2}\,\mathrm{d}\beta} = \frac{[\cos\beta]_{-\alpha}^{\alpha}\,\frac{R^3}{3}}{\alpha R^2} = \frac{2R\sin\alpha}{3\alpha}\,,$$

where the cosine accounts for the distance of the center of mass of each piece from the perpendicular to the axis of symmetry of the slice, ρ denotes the (areal) density of the cake, the base of each piece is $R\,\mathrm{d}\beta$, and its area is therefore $R^2\,\mathrm{d}\beta/2$.

In this way we have derived the position of the center of mass of a homogeneous circular sector. At the moment when the center of mass of a cake slice lies above the edge about which the slice rotates, it is at a height H. This height is determined by the distance of the slice's center of mass from this edge. Clearly, the vertical distance is h, while the horizontal distance is $R_\mathrm{T}\sin\alpha$, which is the perpendicular distance from the center of mass of the sector to one of its edges. We can now write the height H as

$$H = \sqrt{h^2 + \left(\frac{2R\sin^2\alpha}{3\alpha}\right)^2}\,,$$

and the change in potential energy, which equals the work W performed, as

$$\Delta E_\mathrm{p} = W = mg\,(H-h) = \frac{M}{n} g \left(\sqrt{h^2 + \left(\frac{2R\sin^2\alpha}{3\alpha}\right)^2} - h\right) =$$

$$= \frac{M}{n} g \left(\sqrt{h^2 + \left(\frac{2Rn\sin^2\frac{\pi}{n}}{3\pi}\right)^2} - h\right)\,.$$

This relation holds for all $n > 1$. For $n = 1$, i.e. for the entire cake, the position of the center of mass is known (it lies at the center at height h), and the required work is

$$W_1 = Mg\left(\sqrt{R^2+h^2} - h\right)\,.$$

It is worth noting that in this case the cake is no longer rotated about a straight edge as specified in the problem statement, but about its circular rim. One could also consider how much work is required to tip the cake onto its rounded edge. For some values of n this work would be smaller than that required to tip the slice onto a straight edge; however, this possibility is not addressed in the problem.

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