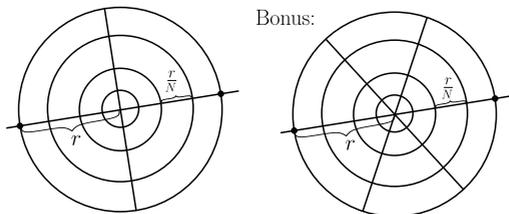


**Problem I.3 ... resisting spiderwebs**

6 points

Consider a spider web formed by a circle with radius  $r$  with two perpendicular crossbars, which go through the middle of the web. We put  $N - 1$  cocentric circles inside the original circle in a certain way, so that the crossbars always intersect at the distance  $r/N$ . How much resistance is between the ends of one crossbar? Material of the web has linear resistivity  $\lambda$  and knots are conductively connected.

Bonus: Determine resistance for three crossbars, which are rotated by  $120^\circ$ .



*Jarda likes animal-themed problems.*

**Main Part of the Problem**

The key to solving this problem is symmetry. If we connect the source to one of the two crossbars of the web, the other crossbar will be perpendicular to it. We will show that in such a case, symmetry implies that on the perpendicular crossbar, there will be zero voltage, and thus zero current. We can therefore neglect it in the solution.

Let us number the circles from the center with index  $n$ . For the inner circle, we have  $n = 1$ , and for the outer circle we have  $n = N$ . Then, we define a web cell as the space bounded by two neighboring circles and the two crossbars that intersect the circles. Furthermore, let us introduce a convenient coordinate system. Its origin shall coincide with the center of the web, the crossbar connected to the source shall coincide with the  $x$ -axis, and the crossbar perpendicular to it shall coincide with the  $y$ -axis. WLOG<sup>1</sup> we can choose any cell of the web (in any quadrant), for instance, the upper left quadrant. Inside the loop, Kirchhoff's second law holds, which states

$$U_{n+1} + U_{n+1,y} - U_n - U_{n+1,x} = 0, \tag{1}$$

where  $U_n$  and  $U_{n+1}$  are the voltages on the parts of the circles  $n$  and  $n + 1$  delimited by the left quadrant.  $U_{n+1,y}$  is the voltage on the segment  $y$  bounded by the circles  $n$  and  $n + 1$ .

Now let us write Kirchhoff's second law for the loop between the  $n$ -th and  $n + 1$ -th circles in the right quadrant. We have

$$V_{n+1} - V_{n+1,x} - V_n - U_{n+1,y} = 0. \tag{2}$$

The situation is symmetric, and the  $x$  crossbar is divided into equal parts with equal resistances, which are of course in series, so all voltages  $U_{n,x}$  and  $V_{n,x}$  will have the same magnitude.

<sup>1</sup>without loss of generality

Therefore  $U_{n,x} = V_{n,x}$ . The same argument applies to the arcs, so  $U_n = V_n$ . Substituting this into equations (1) and (2) and subtracting them, we get

$$0 = U_{n+1} + U_{n+1,y} - U_n - U_{n+1,x} - U_{n+1} + U_{n+1,y} + U_n + U_{n+1,x} .$$

From this, we deduce that

$$2U_{n+1,y} = 0 \quad \Rightarrow \quad U_{n+1,y} = 0 .$$

The resistance of one segment on the  $x$  or  $y$  axis is denoted as  $\rho$ . We have

$$I_{n,y} = \frac{U_{n,y}}{\rho} = 0 .$$

We see that no current flows along the vertical axis, thus we do not need to include it in our circuit.

Let us denote the resistance of one half of the  $n$ -th circle as  $\tilde{R}_n$ . Using rules for combining resistances, we can construct a recurrence relation. Denoting the resistance of the entire web as  $R_N$ , the resistance of a web with  $N - 1$  circles is denoted as  $R_{N-1}$ . The resistance  $R_N$  thus represents the resistance of a circuit consisting of three parallel lines. On two of them lie resistances  $\tilde{R}_N$ , and on the third we have a resistance of  $2\rho + R_{N-1}$ . We obtain the recurrence relation

$$\frac{1}{R_N} = \frac{2}{\tilde{R}_N} + \frac{1}{2\rho + R_{N-1}} \quad \Rightarrow \quad R_N = \frac{1}{\frac{2}{\tilde{R}_N} + \frac{1}{2\rho + R_{N-1}}} . \quad (3)$$

This relation can be iterated down to the resistance  $R_1$ , which is then expressed as

$$R_N = \frac{1}{\frac{2}{\tilde{R}_N} + \frac{1}{2\rho + \frac{1}{\frac{2}{\tilde{R}_{N-1}} + \frac{1}{2\rho + \dots + \frac{1}{2\rho + R_1}}}}} . \quad (4)$$

Next, let's express the resistances  $\rho$ ,  $\tilde{R}_n$ , and  $R_1$ . The resistance  $\rho$  is proportional to  $1/N$  of the radius  $r$ . We get

$$\rho = \lambda \frac{r}{N} . \quad (5)$$

Next, we have the resistance  $\tilde{R}_n$ , which is proportional to the length of one half-arc, so

$$\tilde{R}_n = \lambda \pi r n = \lambda \pi \frac{r}{N} n . \quad (6)$$

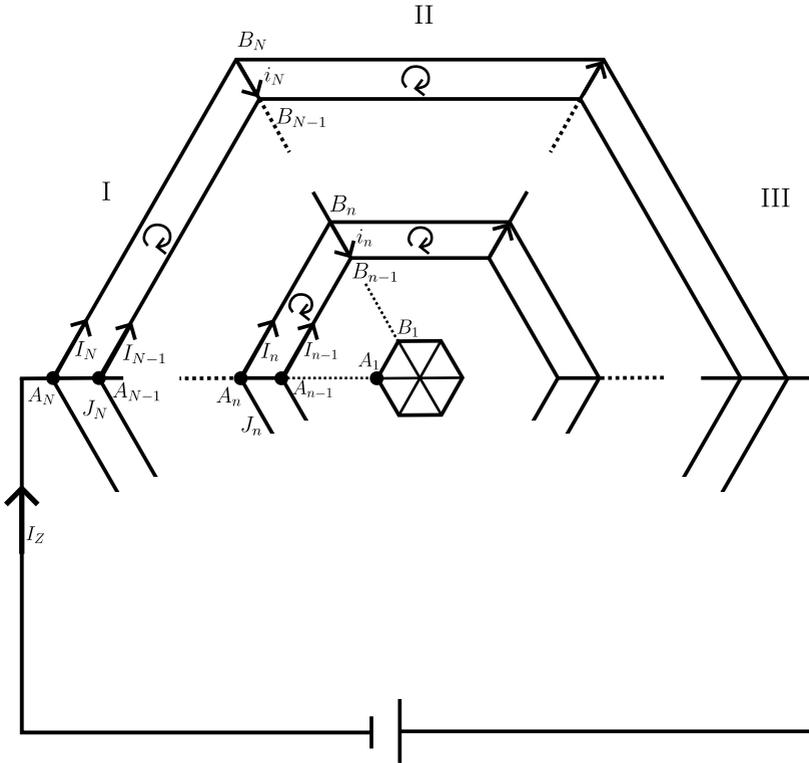
Now we calculate the resistance  $R_1$  as

$$\frac{1}{R_1} = \frac{2}{\tilde{R}_1} + \frac{1}{2\rho} \quad \Rightarrow \quad R_1 = \frac{1}{\frac{2}{\tilde{R}_1} + \frac{1}{2\rho}} \quad \Rightarrow \quad R_1 = \frac{1}{\frac{2N}{\lambda \pi r} + \frac{N}{2\lambda r}} . \quad (7)$$

Substituting relations (5), (6), and (7) into the relation (4) yields the final result

$$R_N = \frac{1}{\frac{2}{\lambda\pi r} + \frac{1}{\frac{2\lambda r}{N} + \frac{1}{\frac{2N}{\lambda\pi r(N-1)} + \frac{1}{\frac{2\lambda r}{N} + \dots + \frac{1}{\frac{2\lambda r}{N} + \frac{1}{\frac{2N}{\lambda\pi r} + \frac{N}{2\lambda r}}}}}}. \quad (8)$$

Bonus



Unfortunately, unlike the previous case, the situation does not simplify enough to be reasonably solvable anymore, so we will have to proceed numerically. In other words, we will construct a system of equations using Kirchhoff's laws, which we will then solve using some computational solver.

Again, let us use symmetries. The web is symmetric along the axis where the source is connected. Now, instead of one transverse axis, we have two axes that form an angle of  $60^\circ$

with the axis to which the source is connected. Let us introduce the following notation. The crossbars divide the upper part of the web into three sectors, labeled from left to right as I, II, and III. The current flowing in the  $n$ -th circle in sector I is denoted as  $I_{n\text{I}}$ , and the current flowing in the same circle in sector III is denoted as  $I_{n\text{III}}$ . Next, let us label the currents along the crossbars. The currents along the horizontal crossbar between circles  $n$  and  $n - 1$  in sectors I and III are denoted as  $J_{n\text{I}}$  and  $J_{n\text{III}}$ , respectively. WLOG, we connect the web to the source along a crossbar, which we will call the horizontal one. Placing the coordinate system at the center of the web, the crossbar dividing sectors I and II in the upper left quadrant is labeled I, and the crossbar dividing sectors II and III in the upper right quadrant is labeled III. The current between the circles  $n$  and  $n - 1$  along these crossbars is denoted  $i_{n\text{I}}$  and  $i_{n\text{III}}$ , respectively. Let us also denote resistances. The resistance along any of the three crossbars intersecting the web between the circles  $n$  and  $n - 1$  is  $\rho$ , calculated as in (5), i.e.,

$$\rho = \lambda \frac{r}{N}.$$

The crossbars divide a circle into six parts. The resistance of one sixth of the  $n$ -th circle is  $\tilde{R}_n$ , given by

$$\tilde{R}_n = \lambda \frac{1}{6} 2\pi r_n = \frac{\pi}{3} \lambda \frac{r}{N} n. \quad (9)$$

The whole spider web is symmetric along the horizontal axis running through its center, so the currents in the upper part are symmetric with those in the lower part. The total current entering and exiting the web must be equal. Let  $I_z$  be the source current. Since the web is symmetric along the vertical axis, the currents exiting each circle must be equal, therefore

$$I_{n\text{I}} = I_{n\text{III}} \equiv I_n J_{n\text{I}} = J_{n\text{III}} \equiv J_n.$$

As the pairs of currents are equal, we labeled them with a single symbol.

Now, let us consider any two cells in sectors I and III and write Kirchhoff's second law

$$\tilde{R}_n I_n + \rho i_{n\text{I}} - \tilde{R}_n I_{(n-1)} - \rho J_n = 0, \quad (10)$$

$$\tilde{R}_n I_n - \rho i_{n\text{III}} - \tilde{R}_n I_{(n-1)} - \rho J_n = 0. \quad (11)$$

Subtracting equations (10) and (11) gives

$$i_{n\text{I}} = -i_{n\text{III}}.$$

We see that currents flowing through the diagonal crossbars have the same magnitude but opposite direction.

Let us now assemble the equations. Points where the horizontal axis intersects the circles are labeled from the center out (excluding the center itself) as  $A_1, A_2, \dots, A_n, \dots, A_N$ . The intersections of axis I with the circles are labeled from the center (again excluding the center) as  $B_1, B_2, \dots, B_n, \dots, B_N$ . Due to symmetry along both axes, the information needed to solve the problem is contained WLOG in a single quarter of the web, e.g., the upper left quadrant. We can now write the equations valid for any  $n$ , however, we need to handle the special cases  $n = 1$  and  $n = N$  separately. For  $n = 1$ , the cells of the circle  $n = 1$  are surrounded by 3 wires instead of 4, unlike the others. For  $n = N$ , no currents  $i_{N+1}$  enter the cells, so they differ from the others.



through the web. For example, we can choose the simplest path, which is the straight path along the horizontal crossbar. We can thus calculate the resistance of this spider web as

$$R_N = \frac{2\rho \sum_{i=1}^N J_i}{I_z} .$$

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