Problem I.1 ... water goblin   3 points

A water goblin lives at the bottom of the Dead Sea. He, as is common for such a folklore creature, traps the souls of people who have drowned in the sea, keeping them in small mugs.

However, the jars are quite small and therefore cannot contain the souls, he has to reduce the volume five-fold by an isothermic process.

One day, the goblin got bored of the Dead Sea and decided to move to the bottom of the Orlík reservoir in southern Bohemia. What is the minimum depth he has to live in at his new place such that the souls are not able to escape by lifting the physical lids?

Before moving, the water goblin used to live at the depth of 33 m such that ascending by any amount would allow the jars to open. Consider the souls to be an ideal gas with the same temperature as that of the surrounding water. At the moment of drowning, just before their compression, the souls are under atmospheric pressure.

The water temperature of the Dead Sea equals 23 °C, water temperature of the Orlík reservoir equals 6 °C, density of the Dead sea equals 1.24 g·cm^{-3}, while the density of the Orlík reservoir equals 0.99 g·cm^{-3}. Consider water properties to be constant in both bodies of water. The jars were properly sealed for the duration of moving from the Dead Sea to the Orlík reservoir.

Terka would like to go to a vacation by the sea from which she will be able to return alive.

At the beginning, the water goblin compresses the little souls into the mugs isothermally, meaning the entire process occurs at the temperature of the Dead Sea, $T_1 = 23\,°C$. The little souls, modeled as an ideal gas at normal atmospheric pressure $p_0 = 101\,325\,\text{Pa}$, must be compressed *fivefold* to fit into the mugs so that the volume of each soul equals the mug's volume V. For the states of the soul before and after compression, Boyle—Mariotte's law for an isothermal process can be written as

$$5Vp_0 = Vp_1\,,$$

from which the pressure of the soul inside the mug p_1 is determined as

$$p_1 = 5p_0 \doteq 0.51\,\text{MPa}\,. \tag{1}$$

Since even a slight lifting of the mugs from a depth of $h_1 = 33\,\text{m}$ in the Dead Sea would cause their lids to open, we can assume that at height h_1 the net force acting on the lid is zero. The cross-sectional area of the lid S is clearly the same on both sides, and by the definition of pressure, it follows that the pressure on both sides of the lid must be equal.

From the inside, the lid is acted upon by the pressure of the soul, p_1, while the external pressure results from Earth's gravitational influence. The first component of the external pressure is the pressure due to the lid's own weight, p_v, arising from its mass m. The lid is also acted upon by the surrounding water with hydrostatic pressure p_{h1} and, finally, by the atmospheric pressure p_0. Altogether, for the Dead Sea we obtain

$$p_1 = p_v + p_{h1} + p_0\,. \tag{2}$$

The lid is subjected to a gravitational force F_G, which gives it the standard gravitational acceleration $g \doteq 9.81\,\text{m·s}^{-2}$. The corresponding pressure on the lid is then

$$p_v = \frac{F_G}{S} = \frac{mg}{S}\,. \tag{3}$$

For the hydrostatic pressure p_{h1} of the liquid at depth h_1, we have

$$p_{h1} = \rho_1 g h_1 \,, \tag{4}$$

where $\rho_1 = 1.24 \, \text{g·cm}^{-3}$ is the density of the liquid—in this case, the water of the Dead Sea. Substituting from equations (1), (3), and (4) into equation (2), we obtain

$$5p_0 = \frac{mg}{S} + \rho_1 g h_1 + p_0 \,. \tag{5}$$

After moving to the Orlík reservoir, we seek the *minimum* depth at which the water goblin's mugs will remain closed. This corresponds to the same condition as in the Dead Sea, where the lids were also just barely closed. Again, we assume a zero net force on the lid, and for the state in Orlík we can write an equation analogous to the equality of pressures (2):

$$p_2 = p_\text{v} + p_{h2} + p_0 \,, \tag{6}$$

where p_2 is the pressure of the soul inside the mug in Orlík and p_{h2} is the hydrostatic pressure at the sought depth h_2 in the reservoir. The atmospheric pressure p_0 and the "self-pressure" of the lid p_v remain unchanged after relocation. However, the water in Orlík is colder, with a temperature of only $T_2 = 6\,^\circ\text{C}$. Since the mug's volume V does not change, an isochoric cooling occurs, which changes the pressure to p_2. According to Charles's law for an isochoric process, we have

$$\frac{p_1}{T_1} = \frac{p_2}{T_2} \quad \Rightarrow \quad p_2 = p_1 \frac{T_2}{T_1} = 5p_0 \frac{T_2}{T_1} \,.$$

Finally, the hydrostatic pressure p_{h2} at depth h_2 in Orlík is determined analogously to equation (4) as

$$p_{h2} = \rho_2 g h_2$$

where $\rho_2 = 0.99 \, \text{g·cm}^{-3}$ is the density of the reservoir water. Substituting the derived expressions back into the second pressure equality (7), we obtain

$$5p_0 \frac{T_2}{T_1} = \frac{mg}{S} + \rho_2 g h_2 + p_0 \,. \tag{7}$$

We now solve the system of equations (5) and (7). At first glance, it contains three unknowns: the mass of the lid m, its cross-sectional area S, and the desired minimum depth h_2 in the Orlík reservoir. Since we need only the last one, we can treat the term $p_\text{v} = mg/S$ as a single unknown, reducing the problem to a system of two equations with two unknowns, which has a unique solution. Expressing the fraction mg/S from equation (5) gives

$$\frac{mg}{S} = 4p_0 - \rho_1 g h_1$$

and substituting this result into equation (7) yields

$$5p_0 \frac{T_2}{T_1} = 4p_0 - \rho_1 g h_1 + \rho_2 g h_2 + p_0 \,.$$

Finally, we solve for and compute the depth h_2 at which the water goblin must settle:

$$h_2 = \frac{1}{\rho_2 g} \left(5p_0 \left(\frac{T_2}{T_1} - 1 \right) + \rho_1 g h_1 \right) \doteq 38 \, \text{m} \,.$$

It is important to note that when performing numerical evaluation, one must use the absolute (thermodynamic) temperature instead of degrees Celsius. For simplicity, the same values of gravitational acceleration g and atmospheric pressure p_0 were assumed for both locations.

Erratum—comment on an error in the original problem statement

The problem initially appeared online with the depth of the water goblin's mugs in the Dead Sea given as $h_1 = 50\,\mathrm{m}$. Although applying the same procedure would yield a result of approximately $60\,\mathrm{m}$, such a problem formulation was physically incorrect. Careful contestants might have noticed, upon back-substitution, that the fraction mg/S came out negative.

We apologize for the error and any confusion it may have caused. The change of the parameter in the problem statement has been taken into account during solution evaluation. We also wish to thank the attentive contestants who noticed and reported this mistake.

Patrik Stercz
patrik.stercz@fykos.org