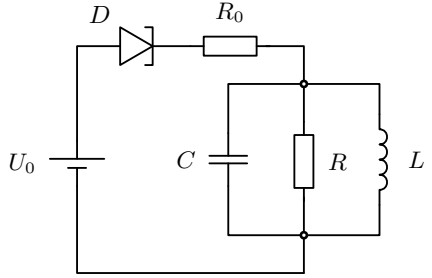


Problem VI.5 ... DC-RLC

9 points

Believe it or not, there are components for which, under certain conditions, the current can decrease as the voltage increases. Consider a diode where, in a voltage interval between U_1 and U_2 (i.e., $U_1 < U_2$), the current decreases linearly from I_1 to I_2 (all values are positive and $I_1 > I_2$). This diode is connected in a series with a resistor of resistance R_0 and a parallel RLC circuit (a resistor R , capacitor C , and inductor L). This circuit is then connected to a DC voltage source U_0 . Determine the current that flows through the circuit. However, when this state is slightly perturbed, we can observe harmonic oscillations. What conditions must be satisfied, and what is the frequency of these oscillations for this to occur? Assume that the voltage and current on the diode are always within the intervals specified above.



Jarda heard about a Nobel Prize awarded for negative resistance.

According to the problem statement, we should first focus on the state where nothing is oscillating. In the parallel branch, current flows through an ideal inductor with zero resistance. Elsewhere in the circuit, voltage and current satisfy the equation

$$U_0 = R_0 I + U_D,$$

here we set $I = I_D$, and the diode voltage U_D and current I_D satisfy a relation that yields an approximately linear current–voltage characteristic (see figure 1)

$$I_D = I_1 - k(U_D - U_1),$$

where $k = (I_1 - I_2)/(U_2 - U_1)$. Substituting U_D into the first equation results in

$$U_0 = R_0 I + \frac{I_1 - I}{k} + U_1.$$

Solving for I yields

$$I = \frac{U_0 - U_1 - \frac{I_1}{k}}{R_0 - \frac{1}{k}}.$$

The values of U_0 and R_0 must be chosen so that the condition $I_2 < I < I_1$ is satisfied. We omit the details, as the problem statement guarantees they hold.

Once the circuit starts oscillating, the situation becomes more complex. The basic idea is that once we lower the voltage on the diode, we increase the current, which leads to an increase of the voltage on the resistors and, therefore, to a decrease of the voltage on the diode. Such a situation would be unstable and the values would stabilize at a point where the direction of the diode volt-ampere characteristic would be negative. The inductance of the coil and the capacity of the capacitor, however, manifest themselves during the change of the current so strongly, that the circuit begins to oscillate harmonically—as we will see in the following calculations.

At first, we will write down the equations describing the time dependence of the voltage and the current, then we will try to guess their solution and we will look for the conditions under

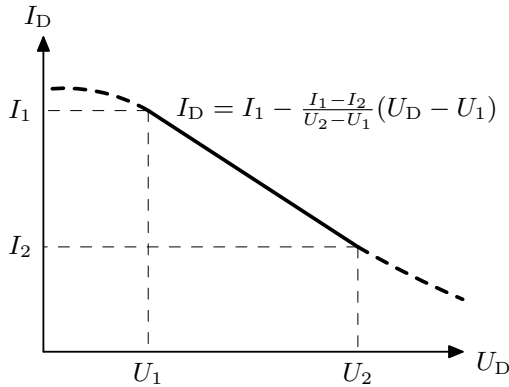


Figure 1: Volt-ampere characteristic of the diode in its linear area.

which this solution applies. We will designate the current in the resistor R_0 , in the diode and in the power source U_0 as I . In the parallel part this current divides itself into three parts flowing through three branches of the circuit. The source voltage U_0 , on the other hand, divides itself into a sum of the voltage on the parallel part, on the resistor $R_0 I$ and on the diode, which we will designate as U_D . In the parallel part of the circuit, the voltage on all the components is equal to

$$U_p = U_0 - R_0 I - U_D .$$

The current flowing through the resistor is equal to $I_R = \frac{U_p}{R}$. For a capacitor, a general relation between its charge Q_C , its capacity C and the voltage on it U_C , written as $Q_C = C U_C$ holds true. If we derive this equation by time, we get an equation

$$I_C = \frac{dQ_C}{dt} = C \frac{dU_C}{dt} ,$$

describing the dependence of the current flowing through the capacitor on voltage. The last component in the parallel part is the coil, for which have the equation

$$U_L = -L \frac{dI_L}{dt} ,$$

where U_L is the voltage on the coil and I_L is the current flowing through the coil. Obviously, the equation $I = I_C + I_L + I_R$ holds true, because no current is created at the division between the parts nor is any current lost anywhere. We can also derive this equation and substitute from the previous expressions. Let us start by writing down

$$\frac{dI}{dt} = \frac{d(I_C + I_L + I_R)}{dt} = \frac{dI_C}{dt} + \frac{dI_R}{dt} + \frac{dI_L}{dt} .$$

Now let us substitute in the individual equations from the previous expressions. In the case of the capacitor and the coil, we also have to derive them by time. We get

$$\frac{dI}{dt} = C \frac{d^2 U_C}{dt^2} + \frac{1}{R} \frac{dU_R}{dt} - \frac{U_L}{L} .$$

As we have stated before, in our case $U_C = U_R = U_L = U_p = U_0 - R_0 I - U_D$. The first time derivative of this expression is

$$\frac{dU_p}{dt} = -R_0 \frac{dI}{dt} - \frac{dU_D}{dt},$$

and the second is

$$\frac{d^2 U_p}{dt^2} = -R_0 \frac{d^2 I}{dt^2} - \frac{d^2 U_D}{dt^2},$$

Therefore, let us substitute into the equation above and we get

$$\begin{aligned} \frac{dI}{dt} &= C \left(-R_0 \frac{d^2 I}{dt^2} - \frac{d^2 U_D}{dt^2} \right) + \frac{1}{R} \left(-R_0 \frac{dI}{dt} - \frac{dU_D}{dt} \right) - \frac{U_0 - R_0 I - U_D}{L} \\ &= -R_0 C \frac{d^2 I}{dt^2} - \frac{R_0}{R} \frac{dI}{dt} - \frac{1}{R} \frac{dU_D}{dt} - C \frac{d^2 U_D}{dt^2} - \frac{U_0}{L} + \frac{R_0}{L} I + \frac{U_D}{L}. \end{aligned}$$

Now we can, according to the problem statement, consider observing harmonic oscillations in the circuit. For the voltage on the diode, we can write

$$U_D = U_s + A \sin \omega t,$$

where A is the amplitude of the oscillations with an angular frequency ω , superimposed with the voltage U_s . The current changes linearly with the voltage on the diode in the considered area, therefore we can describe it as

$$I = I_1 - k(U_D - U_1) = I_1 - kU_s - Ak \sin \omega t + kU_1 = I_s - Ak \sin \omega t,$$

where we have designated $I_1 - kU_s + kU_1$ as I_s . Therefore, we can see that where the voltage on the diode is maximal, the current is minimal and vice versa.

We will now try to substitute these equations for U_D and I into the before-prepared equation for $\frac{dI}{dt}$. Firstly, we will express both of time derivatives

$$\begin{aligned} \frac{dU_D}{dt} &= A\omega \cos \omega t, \quad \frac{dI}{dt} = -kA\omega \cos \omega t, \\ \frac{d^2 U_D}{dt^2} &= -A\omega^2 \sin \omega t, \quad \frac{d^2 I}{dt^2} = kA\omega^2 \sin \omega t. \end{aligned}$$

Now we can substitute for the voltage, the current and their derivatives. After rearranging the terms we get

$$\begin{aligned} -kA\omega \cos \omega t &= -R_0 C k A \omega^2 \sin \omega t + \frac{R_0}{R} k A \omega \cos \omega t - \frac{1}{R} A \omega \cos \omega t \\ &+ C A \omega^2 \sin \omega t - \frac{U_0}{L} + \frac{R_0}{L} I_s - \frac{R_0}{L} A k \sin \omega t + \frac{1}{L} U_s + \frac{1}{L} A \sin \omega t. \end{aligned}$$

This equation has to hold true in every time, therefore the absolute terms on both sides of the equation and the terms multiplied by both the sine and the cosine have to be equal. Due to this, we get three equations

$$\begin{aligned} 0 &= -\frac{U_0}{L} + \frac{R_0}{L} I_s + \frac{1}{L} U_s, \\ 0 &= -R_0 C k A \omega^2 \sin \omega t + C A \omega^2 \sin \omega t - \frac{R_0}{L} A k \sin \omega t + \frac{1}{L} A \sin \omega t, \\ -kA\omega \cos \omega t &= \frac{R_0}{R} k A \omega \cos \omega t - \frac{1}{R} A \omega \cos \omega t. \end{aligned}$$

In the first equation, we can notice the condition calculated in the first part for the direct current operation of the circuit. From the second equation, we get

$$0 = -R_0 CLk\omega^2 + CL\omega^2 - R_0k + 1 \Rightarrow \omega^2 = \frac{1}{LC},$$

after we divide by $A \sin \omega t$, multiply by L and divide by $(1 - R_0k)$, which is a condition for the frequency of alternating current oscillations. This result is not exactly unexpected, it is a well-known equation for the frequency in a regular LC circuit. From the third equation we analogically get

$$1 = (R_0 + R)k.$$

It is therefore obvious that for the harmonic oscillations to appear in the circuit, a specific choice of resistors is necessary in the circuit based on the properties of the diode, given by the value of k . We can also notice that if the current flowing through the diode grew with an increase in voltage (e.g. if k would be negative, according to our definition), resistors with negative resistance would be necessary to generate harmonic oscillations.

The oscillation frequency $\omega = \sqrt{1/LC}$ therefore does not depend on the values of the resistors resistance. In the end, let us remind you of the value

$$k = \frac{I_1 - I_2}{U_2 - U_1},$$

which we have found by analyzing the volt-ampere characteristic of the diode.

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