Problem VI.3 ... fast and FYKOS

6 points; (chybí statistiky)

In an action move, a car is moving on a road with speed v_0. There is a truck traveling in front of the car at a speed $v_\mathrm{k} < v_0$ with its loading area opened and prepared for the car to drive into. What will be the speed of the car after its spinning wheels get onto the truck?

Assume that the front wheels of the car will be slipping for a very short time after making contact with the truck; we are interested in the speed right after the wheels stop slipping. Each car wheel has a radius r and a moment of inertia I. The car mass is equal to M, while the truck mass is substantially greater. The truck will therefore see no change in speed throughout the process, and its wheels are not slipping. *Lego remembered the Mythbusters.*

When the front wheels enter the moving truck, they start to skid on it. Therefore, a friction force acting between them and the truck will arise. This friction force acts on the car in the forward direction, so it accelerates the car, just as we would expect. However, this same friction force also acts against the rotation of the wheels, so the wheels slow down their rotation. Hence, the car cannot possibly accelerate to the speed $v_0 + v_\mathrm{k}$.

While the car is driving on the road, the peripheral speed of the wheels equals the car's speed, so their angular velocity is $\omega_0 = v_0/r$. Skidding stops when the car's speed v_a equals the sum of the peripheral speed of the front wheels $v_\mathrm{p} = \omega r$ and the truck's speed v_k.

If we denote the friction force between each wheel and the truck as F_t, then the torques acting on the front wheels are $M = F_\mathrm{t} r$, so their angular acceleration is $\varepsilon = M/J = F_\mathrm{t} r/J$. After a skidding time t, their angular velocity is

$$\omega(t) = \omega_0 - \varepsilon t = \omega_0 - \frac{tF_\mathrm{t} r}{J}\,,$$

and their peripheral speed is

$$v_\mathrm{p}(t) = \omega(t) r = \omega_0 r - \frac{tF_\mathrm{t} r^2}{J} = v_0 - \frac{tF_\mathrm{t} r^2}{J}\,.$$

The speed of the car itself is more complicated. It is accelerated by a force $2F_\mathrm{t}$. But besides the front wheels, the car also has rear wheels, and these must not skid on the road—they must rotate faster as the car accelerates. This angular acceleration is driven by the friction between the rear wheels and the road, but the same friction also slows down the car. Let's assume the car accelerates with acceleration a. In that case, the rear wheels must have angular acceleration $\varepsilon_\mathrm{z} = a\ /r$, and must therefore be acted on by a torque $M_\mathrm{z} = \varepsilon_\mathrm{z} J$, meaning the friction force between them and the road is $F_\mathrm{tz} = M_\mathrm{z}/r = \varepsilon_\mathrm{z} J/r = a\ J/r^2$. Thus, we can write the equation for the acceleration of the car and manipulate it

$$\begin{aligned}
F_\mathrm{celk} &= Ma\,,\\
2F_\mathrm{t} - 2F_\mathrm{tz} &= Ma\,,\\
F_\mathrm{t} - a\ \frac{J}{r^2} &= \frac{1}{2}Ma\,,\\
\frac{F_\mathrm{t}}{\frac{J}{r^2} + \frac{M}{2}} &= a\,.
\end{aligned}$$

After the skidding time t, the speed of the car is

$$v_\mathrm{a}(t) = v_0 + a\ t = v_0 + \frac{tF_\mathrm{t}}{\frac{J}{r^2} + \frac{M}{2}}\,.$$

Note that in the final expressions for the speed of the car and the wheels, the friction force F_t and the skidding time t appear only as the product tF_t. We are not interested in their exact values, so we can substitute this product as $tF_\mathrm{t} = I$, where this quantity is called the impulse.

Let's substitute into the no-skidding condition

$$
\begin{aligned}
v_\mathrm{p} + v_\mathrm{k} &= v_\mathrm{a} \\
v_0 - \frac{I\ r^2}{J} + v_\mathrm{k} &= v_0 + \frac{I}{\frac{J}{r^2} + \frac{M}{2}}\,, \\
v_\mathrm{k} &= \frac{I\ r^2}{J} + \frac{I}{\frac{J}{r^2} + \frac{M}{2}}\,, \\
I &= \frac{v_\mathrm{k}}{\frac{r^2}{J} + \frac{1}{\frac{J}{r^2} + \frac{M}{2}}}\,,
\end{aligned}
$$

and thus, we obtain the final force impulse after which the front wheels stop skidding. At that moment, the speed of the car is

$$
\begin{aligned}
v_\mathrm{a} &= v_0 + \frac{I}{\frac{J}{r^2} + \frac{M}{2}}\,, \\
v_\mathrm{a} &= v_0 + \frac{v_\mathrm{k}}{\frac{r^2}{J} + \frac{1}{\frac{J}{r^2} + \frac{M}{2}}} \cdot \frac{1}{\frac{J}{r^2} + \frac{M}{2}}\,, \\
v_\mathrm{a} &= v_0 + \frac{v_\mathrm{k}}{\frac{r^2}{J}\left(\frac{J}{r^2} + \frac{M}{2}\right) + 1}\,, \\
v_\mathrm{a} &= v_0 + \frac{v_\mathrm{k}}{2 + \frac{r^2}{J} \cdot \frac{M}{2}}\,.
\end{aligned}
$$

In other words, the speed of the car increases by a term proportional to the speed of the truck. In the denominator, we see the number 2—even if the car had no mass, there would still be rear wheels besides the front ones. The second term in the denominator tells us how many times the mass of the car is more significant than the moment of inertia of the wheels.

If the wheel were a solid cylinder of mass m, then $J/r^2 = m/2$ would hold. The car would then accelerate by a factor of $v_\mathrm{k}/(2 + M/m)$, where the 2 in the denominator would likely be negligible. It becomes clear that the heavier the wheels are, the less likely it is for them to change their rotational speed, the car therefore has greater acceleration. Conversely, the heavier the car is, the less it accelerates. We see the car does accelerate, but since it is easier to change the rotational speed of the wheels than the linear speed of the whole car, it does not accelerate much.

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