## Problem VI.1 ... Planck pencil

3 points; (chybí statistiky)

*What is the shortest possible length a pencil can reach through sharpening? The hand can exert a maximum pressure $p$ on the pencil and the pressure is uniformly distributed over the contact area; the sharpening requires a torque $M$. The pencil is cylindrical with diameter $d$, the sharpener has length $h$, and the coefficient of static friction between the hand and the pencil is $f$. You may assume that the pencil can always be gripped optimally. Consider normal sizes of both hands and pencils only. Sharpened pencil can be defined as one with a conical tip.*

Bonus: *What is the maximum achievable efficiency of writing with a pencil? By efficiency, we mean the fraction of the pencil's graphite volume actually used for writing. The graphite core initially has a sharp conical tip; if not sharpened, the tip is a cylinder with height $l$ and radius $R$. The sharpener removes material so that the tip of the pencil is conical after sharpening. Assume the pencil to always be perpendicular to the paper while writing.*

*Marek J. prefers triangular pencils.*

Our experience with sharpening pencils is that it is easy at first, but as the pencil gets shorter, sharpening becomes more and more difficult. This is because as the pencil shortens, we have less surface area available to grip and rotate the pencil for sharpening. Typically, at the beginning, we can use the entire contact area with our hand, but over time, this contact area decreases.

From the problem statement, we know that sharpening requires applying a torque of at least $M$. By definition, torque is given by

$$M = Fx\,,$$

where $x$ is the perpendicular distance from the axis of rotation—in our case, this is half the diameter of the pencil, $x = d/2$. $F$ is the frictional force from our hand grip, which we use to try to rotate the pencil in the sharpener. It is given by a simple relation $F = F_\mathrm{n} f$, where $F_\mathrm{n}$ is the normal force and $f$ is the coefficient of friction given in the problem statement. Next, we need to express what normal force $F_\mathrm{n}$ we can exert on the surface of the pencil, knowing that the maximum pressure from our grip is $p$. We use the relation $F_\mathrm{n} = pS$, where $S$ is the area over which we apply the pressure. In our case, we calculate the cylindrical surface of the pencil as $S = \pi dl$, where $l$ is the length of the pencil *sticking out* of the sharpener. We get the following condition

$$M \leq \frac{\pi p d^2 l f}{2}\,,$$

which confirms our experience that as the pencil length $l$ decreases, we need to grip it more tightly to sharpen it. Since $p$ is already the maximum pressure we can apply, we obtain the minimum length of the protruding part of the pencil

$$l_\mathrm{min} = \frac{2M}{\pi p d^2 f}\,.$$

Including the part inside the sharpener, the shortest possible length of a sharpened pencil is $L$, which is

$$L = l_\mathrm{min} + h = \frac{2M}{\pi p d^2 f} + h\,.$$

*Bonus*: We start with a sharpened pencil lead of volume $V$. The trick is to realize that once the pencil is sharpened such that the radius of the top base of the sharpened cone is only infinitesimally smaller than $R$ (i.e., we have nearly used up that part of the pencil), we can achieve efficiency arbitrarily close to one. Every time we use a tiny bit of the lead for writing, we immediately sharpen it by an infinitesimally small amount—repeating this with infinitesimally small losses of lead during sharpening. Let us assume we have used the lead to the point where the top base has radius $R$ (i.e., the pencil is fully used). Now, we only need to sharpen the corners of the pencil infinitesimally so that the radius of the top base becomes $R-2\varepsilon$. The height of these "triangular corners" also scales with $\varepsilon$, so their surface area is proportional to $\varepsilon^2$. After rotation (around the pencil's axis), this gives a volume proportional to $R\varepsilon^2$. The key point is that if we lose a volume $V_\mathrm{s}$ proportional to $R\varepsilon^2$ during sharpening, the usable lead volume in the conical shape is $V_\mathrm{p}$ proportional to $R^2\varepsilon$. Thus, for efficiency $\eta$, we have

$$\eta = \frac{V_\mathrm{p}}{V} = \frac{V\ \ - V_\mathrm{s}}{V} = 1 - \frac{V_\mathrm{s}}{V_\mathrm{p} + V_\mathrm{s}} \simeq 1 - \frac{R\varepsilon^2}{R^2\varepsilon + R\varepsilon^2}\,,$$

which in the limit $\varepsilon \to 0$ approaches one.

Just like in thermodynamics, processes with maximum efficiency (in our case, writing with 100% lead usage) are infinitely slow.

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