Problem V.5 ... rise, entropy, rise

9 points; průměr 6,79; řešilo 29 studentů

Marek has two identical metal cubes with constant heat capacity C, one at temperature T_1 and the other one at T_2. What is the highest and lowest temperature at which they can both stabilize if he brings them into contact and only uses them to power a heat engine?

Hint: *If you get stuck, remember that entropy will never decrease.*

Marek settled for a thermal equilibrium, if not for the inner one.

the first block as U_1, the second as U_2 and the work of the heat engine as W. We can express the relationship between them

$$\Delta U_1 + \Delta U_2 + W = 0\,.$$

Without loss of generality, we can assume $T_2 > T_1$. If the system stabilizes at temperature T_f, the change in internal energy of the blocks can be expressed through using the heat capacity C as

$$\begin{aligned}\Delta U_1 &= C\left(T_\mathrm{f} - T_1\right)\,,\\ \Delta U_2 &= C\left(T_\mathrm{f} - T_2\right)\,.\end{aligned}$$

Substituting into the equation, we obtain

$$2T_\mathrm{f} - T_1 - T_2 + \frac{W}{C} = 0\,, \tag{1}$$

where T_1, T_2 and C are the values provided; we are solving for T_f—the only changing variable is W, and since W is a non-negative quantity, we can see that the highest attainable value of T_f will be at $W = 0$. The maximum temperature in such a case equals

$$T_\mathrm{max} = \frac{T_1 + T_2}{2}\,,$$

as expected.

What can be said about the minimum temperature? It will certainly be higher than T_1 and will occur when the work is maximum. Here, the fact that the total entropy cannot decrease comes into play:

$$\Delta S \geq 0\,.$$

If we denote the change in entropy of the engine as S_s and the entropy of the first and second block as S_1 and S_2, respectively, we get

$$\Delta S_1 + \Delta S_2 + \Delta S_\mathrm{s} \geq 0\,.$$

An ideal engine will have $\Delta S_\mathrm{s} = 0$. For the blocks, Clausius' relation holds:

$$\mathrm{d}S = \frac{\delta Q}{T} = \frac{1}{T}\frac{\delta Q}{\mathrm{d}T}\,\mathrm{d}T = \frac{C}{T}\,\mathrm{d}T\,.$$

Through integrating for the first block, we obtain

$$\Delta S_1 = \int_{T_1}^{T_\mathrm{f}} \mathrm{d}S = C\int_{T_1}^{T_\mathrm{f}} \frac{1}{T}\,\mathrm{d}T = C\log\frac{T_\mathrm{f}}{T_1}\,.$$

The condition for entropy is then

$$\begin{aligned}\log\frac{T_\mathrm{f}}{T_1}+\log\frac{T_\mathrm{f}}{T_2}&\geq 0\,,\\ \log\frac{T_\mathrm{f}^2}{T_1T_2}&\geq 0\,,\\ \frac{T_\mathrm{f}^2}{T_1T_2}&\geq 1\,,\\ T_\mathrm{f}&\geq\sqrt{T_1T_2}\,.\end{aligned}$$

Is the lower bound of this inequality achievable? Let us return to the energy conservation principle to verify that this solution corresponds to non-negative work. From equation (1), we see that the work will be maximized when T_f is minimized (which makes sense from a physical standpoint). We see that the maximum attainable work equals

$$W_\mathrm{max}=C\left(T_1+T_2-2\sqrt{T_1T_2}\right)=C\left(\sqrt{T_1}-\sqrt{T_2}\right)^2\geq 0\,.$$

To verify, we can substitute into (1) and calculate the minimum temperature

$$T_\mathrm{min}=\frac{T_1+T_2}{2}-\frac{1}{2}\left(\sqrt{T_1}-\sqrt{T_2}\right)^2=\sqrt{T_1T_2}.$$

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