Problem IV.5 ... smoker at a tram stop 10 points; průměr 6,00; řešilo 40 studentů

Jarda is standing at a tram stop, waiting for the tram. However, it still hasn't arrived, so he decides to walk at a speed v toward an information board located dd meters away to check the timetable. Next to the board, someone is passing the time by smoking a cigarette. Determine when Jarda will get close enough to the smoker to smell the smoke. The concentration of smoke particles at a distance $d_0 = 1\,\mathrm{m}$ from the smoker is c_0. Jarda will notice the smell if the concentration of smoke particles at his position reaches c_0/N. Consider the smoker to be a symmetrically spherical emitter of smoke and assume there is no wind.

And the worst ones walk back and forth around the tram stop.

Let us first clarify the physical framework of the problem. The problem states that there is no wind, yet smoke particles spread out into the space. This phenomenon is called *diffusion*—the process of spontaneous net movement, typically from areas of higher concentration to areas of lower concentration. It is similar to when a small amount of dye is added to water—over time, the entire volume of water becomes uniformly colored without the need for stirring.

As mentioned before, when particles are free to move in a space, they spread from areas of higher concentration to areas of lower concentration. The essence of this process is their random thermal motion. Atoms undergo disordered random movement and exchange positions. However, if there are more atoms of a certain type in point A than in point B, more of them will move from point A to point B than the other way around; the probability of transition is the same, but initially, there are more atoms in point A. Thus, substances diffuse from areas of higher concentration to areas of lower concentration and spontaneously try to equalize their concentration throughout the space.

Consider a small area and count the number of particles passing through it in one direction per unit time. Let us denote this as j_1. In the opposite direction, the number of particles passing through this surface is j_2, which may be different. The *particle flux* is defined as the quantity $j = j_1 - j_2$, which is the difference between these numbers.

According to the problem, the situation is stationary, and new smoke particles are continuously generated by the smoker. These particles diffuse in order to balance their concentration throughout the space. The area of the open tram stop is large enough for us to assume that that the particles will not accumulate anywhere and they must eventually travel to infinite distance. Thus, their concentration around the smoker does not change over time. However, if the smoker produces x particles per unit time, all of these particles must effectively pass through some area around him, otherwise, they would accumulate at his location. Since the situation is spherically symmetric, consider a sphere with radius r centered at the smoker as this surface. By symmetry, the particle flux j passing through every point on this sphere is the same, and in order for the concentration not to change, the following has to hold:

$$x = 4\pi r^2 j \quad \Rightarrow \quad j = \frac{x}{4\pi r^2}\,,$$

where $4\pi r^2$ is the surface area of our imaginary sphere. Therefore, the particle flux decreases with distance from the smoker in direct proportion to $1/r^2$.

Let us return to our earlier considerations. We mentioned that the diffusion rate depends on the concentration; particles move from areas of higher concentration to areas of lower concentration. From our basic assumptions, we also deduced that j_1 is proportional to the concentration on one side c_1, and similarly j_2, is proportional to the concentration on the other side c_2. Their difference, i.e., the flux, equals

$$j = j_1 - j_2 = Dc_1 - Dc_2 = D\left(c_1 - c_2\right) ,$$

where the proportionality constant is denoted as D, often called the *diffusion coefficient*. We can see that the flux is proportional to the difference in concentration on either side of the surface. If the concentration varies spatially from point to point, it is possible to define the particle flux as a vector and rewrite the previous equation as *Fick's first law*:

$$\mathbf{j} = -D\nabla c ,$$

where ∇ is the symbol for the *gradient* operator. This operator tells us how quickly the quantity it acts upon changes in the direction of its greatest increase—in essence, it represents the derivative of this quantity with respect to all coordinates. The negative sign indicates that the particle flux occurs from areas of higher concentration to areas of lower concentration, as we mentioned earlier.

We could determine the concentration profile by solving this equation, that is if we are able to express the particle flux as a position function. Generally, solving such an equation is mathematically difficult; we will approach it (perhaps somewhat unexpectedly) by using an analogy with the gravitational field.

Let us consider a spherically symmetric gravitational field as the ones formed around stars. The force F is, according to Newton's law, proportional to $1/r^2$, just like our particle flux. At the same time, we can introduce a potential energy V in this field, which is proportional to $-1/r$. In general, the relationship between force and potential energy is the one you might have learned in school, which is

$$\mathbf{F} = -\nabla V \ .$$

This equation has the same form as Fick's first law, except for a constant factor! Therefore, it will have the same solution. So, if the particle flux is proportional to $1/r^2$ (like the force), the concentration will be proportional to $-1/r$ (like the potential energy)!

If $c \propto 1/r$, then the product $c{\cdot}r = K$ is constant throughout the space. In particular, $c_0{\cdot}d_0 = K$, where we used the values from the problem statement. If the limiting concentration that Jarda senses is $c_{\mathrm{lim}} = c_0/N$, than this will occur when the distance is

$$d_{\mathrm{lim}} = \frac{K}{c_{\mathrm{lim}}} = \frac{K\ N}{c_0} = \frac{c_0 d_0 N}{c_0} = N d_0 \,.$$

Since the problem asks for the time it takes for Jarda to sense the smoker after starting to move, we need to calculate the difference in distances between Nd_0 and Jarda's initial position d and divide it by his speed

$$t = \frac{d - Nd_0}{v} \,.$$

If $d < Nd_0$, Jarda would have already sensed the smoker at the other end of the tram stop, but according to the problem, this did not happen.

We have obtained a very simple relationship. This is of course the result of many simplifications we made along the way. However, we also see the mathematical elegance of physics here. In our solution, we used an analogy with the gravitational force, which might initially seem completely unrelated. This is possible because these phenomena are described by equations of the same form, just with different "letters". The same equations must have the same solution. Consider whether a similar analogy could be made using the laws of electrostatics.

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