Problem IV.4 ... examination time

7 points; (chybí statistiky)

Jarda is preparing for his special relativity exam in his vacation house on one of Jupiter's moons. He was not keeping track of time and found out his exam begins in two hours (his time is synced with Earth). He got into his extremely fast rocket and set out for Earth. At the time of takeoff, the distance between his rocket and Earth equaled 8 AU. He wants to study during his journey, but he's learning at a rate 1.5 times slower compared to when he's sitting in front of the examination room, as he needs to focus on controlling the ship while in flight. At what speed does he need to be flying to learn as much as possible? The ship is flying at a constant speed; do not consider the time required to accelerate and decelerate.

Jarda lives far away in Brno.

Let $T = 2\,\mathrm{hr}$ be the time until the exam as it is perceived by an observer on Earth (He is the one judging Jarda's learning rate). We split this time into the time t_l Jarda spends in his rocket and a t_Z he spends on Earth. Jarda's time passes slower while he travels in his rocket, so he will be learning for the time t_l/γ, where

$$\gamma = \frac{1}{\sqrt{1 - \left(\frac{v}{c}\right)^2}}$$

is the Lorenz factor determining the dilatation time, and v is the speed of the rocket. He can set the speed as he wishes, but $vt_l = d$ has to hold, where $d = 8\,\mathrm{AU}$. He also needs to remember that he has to be quick enough to make it to Earth on time.

Let w be the speed of learning Jarda can achieve on Earth. We need to find the maximum of the formula

$$wt_Z + \frac{w}{n}t_l\,,$$

where $n = 1.5$ is a factor describing the rate at which Jarda learns more slowly while traveling in the rocket. We solve for t_l; after solving, we obtain the function of the amount learned as

$$w\left((T - t_l) + \frac{1}{n}\sqrt{1 - \left(\frac{d}{ct_l}\right)^2}\,t_l\right)\,.$$

By introducing a substitution $x = ct_l/d$, we obtain

$$\frac{wd}{c}\left(\frac{Tc}{d} - x + \frac{1}{n}\sqrt{1 - \frac{1}{x^2}}\,x\right)\,.$$

Derivative of this function with respect to x is equal to

$$\frac{wd}{c}\left(-1 + \frac{1}{n}\frac{\frac{1}{x^2}}{\sqrt{1 - \frac{1}{x^2}}} + \frac{1}{n}\sqrt{1 - \frac{1}{x^2}}\right)\,,$$

from which we can calculate the value for which the derivative is equal to zero by solving

$$n\sqrt{1 - \frac{1}{x^2}} = 1\,,$$

which is equal to

$$x = \sqrt{\frac{1}{1 - \frac{1}{n^2}}}\,.$$

The optimal rocket speed is then

$$v = \frac{c}{x} = \sqrt{1 - \frac{1}{n^2}}c = \frac{\sqrt{5}}{3}c \doteq 0.75c\,.$$

To verify that this is, in fact, the speed needed for Jarda to learn the most, we can either calculate the second derivative ourselves or plot the derivative of the original function in appropriate graphing software.

The last thing we need to verify is whether the speed is sufficient for reaching the Earth in time as the calculated formula does not contain the time T, nor does it contain the distance d. The time spent in the rocket is equal to

$$t_\mathrm{l} = \frac{d}{v} \doteq 90\,\mathrm{min}\,,$$

meaning Jarda will make it in time just fine.

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