Problem IV.3 ... a sphere in a shadow
5 points; průměr 4,11; řešilo 64 studentů

A small sphere is positioned at the furthest possible distance from Earth such that that it remains completely within the planet's shadow. At what temperature will the sphere reach equilibrium, assuming Earth behaves as a black body with a homogeneous temperature of $T_\mathrm{E} = 20\,\mathrm{^\circ C}$? Neglect all light sources other than the Sun and assume that light rays propagate in straight lines, ignoring refraction in the atmosphere or relativistic effects.

One of Jonáš's marbles rolled all the way under the bed.

The Earth will cast a shadow in the shape of a cone, whose dimensions will primarily depend on the radii of the Earth and the Sun—denoted as $R_\mathrm{Z} = 6.38 \cdot 10^3\,\mathrm{km}$ and $R_\mathrm{S} = 6.96 \cdot 10^5\,\mathrm{km}$, respectively—and the distance between the Sun and the Earth $d_\mathrm{Z} = 1\,\mathrm{AU} = 1.50 \cdot 10^8\,\mathrm{km}$. Another question posed by the problem statement is about determining the temperature at the tip of this shadow. We will begin by deriving its dimensions. Due to symmetry, we can reduce the entire situation to a plane.

Let us denote the length of the shadow as D. The furthest point of the shadow, by symmetry, will lie on the line connecting the centers of the Earth and the Sun. At the same time, this point must also lie on the common tangent between the Earth and the Sun. Any closer point would require a ray to pass through the Earth, which is physically impossible. Intuitively, this common tangent will be nearly parallel to the line connecting the centers of the planet and the star. The situation is illustrated in figure 1.

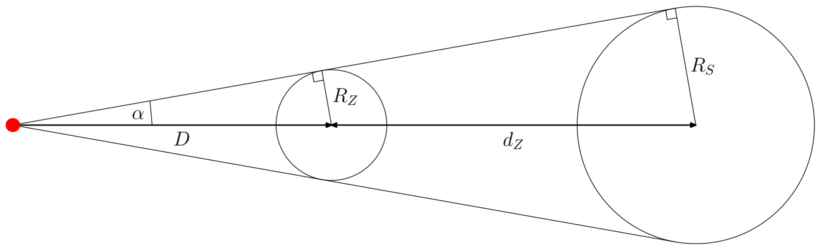

Figure 1: Diagram of the situation.

For reference, the angle between the common tangent and the centerline is $\alpha \approx \sin\alpha = R_\mathrm{Z}/D \doteq 0.005$, which allows us to use this approximation. The calculations for the length of the shadow D, based on the triangle similarities, are as follows

$$\sin\alpha = \frac{R_\mathrm{Z}}{D} = \frac{R_\mathrm{S}}{D + d_\mathrm{Z}}\,,$$

$$D = \frac{d_\mathrm{Z}}{\frac{R_\mathrm{S}}{R_\mathrm{Z}} - 1} = d_\mathrm{Z}\frac{R_\mathrm{Z}}{R_\mathrm{S} - R_\mathrm{Z}}\,.$$

Substituting the values, we obtain $D \doteq 1.4 \cdot 10^6\,\mathrm{km} \doteq 9.3 \cdot 10^{-3}\,\mathrm{AU}$.

The small sphere will absorb heat radiated by the Earth (but not by the Sun, since it is in a shadow!), and it will also radiate energy according to the Stefan-Boltzmann law. In thermal equilibrium, these processes balance each other. The energy received per unit time from the Earth's radiation can be calculated as

$$P_\mathrm{in} = A_\mathrm{k}\frac{L_\mathrm{Z}}{4\pi D^2} = A_\mathrm{k}\frac{\sigma S_\mathrm{Z} T_\mathrm{Z}^4}{4\pi D^2} = \sigma A_\mathrm{k} T_\mathrm{Z}^4 \left(\frac{R_\mathrm{Z}}{D}\right)^2\,,$$

where $T_\mathrm{Z} = 20\,^\circ\mathrm{C}$ is the Earth's temperature, S_Z and L_Z are Earth's surface area and total luminosity, $\sigma = 5.670\,\mathrm{W\cdot m^{-2}\cdot K^{-4}}$ is the Stefan-Boltzmann constant, and A_k is the surface area of the sphere receiving radiation. In our case, we use the common approximation $A_\mathrm{k} \approx \pi r_\mathrm{k}^2$ with r_k as the sphere's radius. This can be justified by the the sphere being very small and the rays arriving nearly parallel, so the sphere receives the same radiation as a disk of radius r_k. On the other hand, the sphere radiates energy at a rate of

$$P_\mathrm{out} = \sigma S_\mathrm{k} T_\mathrm{k}^4 = 4\pi\sigma r_\mathrm{k}^2 T_\mathrm{k}^4\,.$$

Equating the two expressions, we can solve for the equilibrium temperature T_k

$$\begin{aligned} P_\mathrm{out} &= P_\mathrm{in}\,,\\ 4\pi\sigma r_\mathrm{k}^2 T_\mathrm{k}^4 &= \pi\sigma r_\mathrm{k}^2 T_\mathrm{Z}^4 \left(\frac{R_\mathrm{Z}}{D}\right)^2\,,\\ T_\mathrm{k} &= T_\mathrm{Z}\sqrt{\frac{R_\mathrm{Z}}{2D}} = T_\mathrm{Z}\sqrt{\frac{R_\mathrm{S} - R_\mathrm{Z}}{2d_\mathrm{Z}}}\,. \end{aligned}$$

The resulting temperature is then $T_\mathrm{k} \doteq 14\,\mathrm{K}$.

Jonáš Dej
jonas.dej@fykos.org

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