Problem III.4 ... forever Young

7 points; průměr 4,29; řešilo 66 studentů

Marek has a double slit with negligible slit width, immense slit height; the distance between the slits equals b. Light of wavelength λ is incident on the slit. A nearby screen on which the interference pattern is formed is moving away from the double slit at a small velocity v. What is the velocity of the m-th order maximum on the screen? *Marek sympathized with the Master.*

Let us denote the distance of the double slit from the screen as x. If we encounter the double slit experiment in a high school physics textbook, we will usually see the formula

$$\Delta y = \frac{x}{b}\lambda\,,$$

where Δy denotes the distance between two secondary interference maxima. However, this relation holds only for large distances, so it cannot be used for solving this problem; however, let us keep this formula in mind, as at the end we will verify that for the maximum of m-th order, we do actually get the velocity as

$$\widetilde{v_y} = m \cdot \frac{v}{b}\lambda\,.$$

To find the solution to the problem of a close screen, let us first think about the geometry of the whole situation. The interference maximum occurs precisely when the path difference between the incident rays is equal to an integer multiple of the wavelength λ, so for the maximum of the m-th order, it will be a difference $m\lambda$. If we change the position of the screen and follow the curve along which the points of the m-th order maximum move in space, we will find that it is a hyperbola: by definition, a hyperbola is a set of points that have a constant difference in distance from two foci. So let us put the screen aside for a moment and focus on what this hyperbola looks like.

Consider a coordinate system in which the slits are located at the points $[0,\,-b/2]$ and $[0,\,b/2]$. We are interested in what will the hyperbola look like. As per the definition stated previously, this hyperbola is a set of points that have a constant distance of $m\lambda$ from the individual slits. From the geometry of conics, we know that we are looking for a hyperbola with a semi-major axis (half of the path difference) of length $m\lambda/2$ and the eccentricity (distance between the foci and the center) $b/2$. This hyperbola is then described by the equation

$$\frac{4y^2}{m^2\lambda^2} - \frac{4x^2}{b^2 - m^2\lambda^2} = 1\,.$$

We need to realize what we are solving for. The screen is moving away at a velocity v in the direction of the horizontal axis, the velocity is therefore equal to the time derivative $v = \frac{\mathrm{d}x}{\mathrm{d}t}$. The velocity of the maximum of the m-th order will then be equal to $v_y = \frac{\mathrm{d}y}{\mathrm{d}t}$, where the point $[x,\,y]$ lies on our hyperbola. To determine v_y, we can differentiate the equation of the hyperbola with respect to time, while considering x and y as functions of time (and thus differentiating everything as composite functions). We get

$$\frac{8y}{m^2\lambda^2}\frac{\mathrm{d}y}{\mathrm{d}t} - \frac{8x}{b^2 - m^2\lambda^2}\frac{\mathrm{d}x}{\mathrm{d}t} = 0 \quad\Rightarrow\quad v_y = \frac{x}{y}\frac{m^2\lambda^2}{b^2 - m^2\lambda^2}v\,.$$

We solve the hyperbola equation for y, obtaining

$$y = \pm\frac{m\lambda}{2}\sqrt{1 + \frac{4x^2}{b^2 - m^2\lambda^2}}\,.$$

We obtain two values here because at each point x there are two points of an interference maximum. Furthermore, because of symmetry and simplicity, it is sufficient to consider only the positive value. Let us substitute this result back into the differentiated equation and solve it for v_y. We obtain the formula

$$v_y = \frac{2m\lambda x}{\sqrt{(b^2 - m^2\lambda^2 + 4x^2)(b^2 - m^2\lambda^2)}} v\,,$$

which is the solution to our problem. We see that the speed of the interference maximum depends not only on the expected parameters, but also on the distance from the double slit.

For a somewhat more compact notation, we can include the assumption that $b \gg \lambda$ holds—in real situations, the wavelength of the radiation is often negligible compared to the distance between the slits—we can use the approximation

$$\frac{1}{\sqrt{1 - \frac{m^2\lambda^2}{b^2}}} \approx 1 + \frac{m^2\lambda^2}{2b^2} \approx 1\,,$$

which gives us a slightly simpler result

$$v_y \approx \frac{2m\lambda x}{b\sqrt{b^2 + 4x^2}} v\,.$$

Finally, if we consider—in spite of the assumptions from the problem statement—that the double slit is far away and that $x \gg b \gg \lambda$ holds, we can approximate

$$\frac{1}{\sqrt{1 + \frac{b^2 - m^2\lambda^2}{4x^2}}} \approx 1 - \frac{b^2 - m^2\lambda^2}{8x^2} \approx 1\,,$$

which gives us

$$v_y \approx m\frac{v}{b}\lambda = \widetilde{v_y}\,,$$

which corresponds to the formula derived by the "naive" approach at the beginning.

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