Problem III.3 ... non-coloumbic 6 points

Fykosaurus discovered a previously unknown type of interaction while being in a lab. He found a small spherical object in a dusted cabinet. When he placed a point mass with a mass m and released it, the point mass always collided with the sphere after a specific time t. Determine the force by which the mass point is attracted to the unknown object as a function of their mutual distance. Consider that everything takes place on a horizontal plane without resistive forces in the framework of classical mechanics. In addition, the Fykosaurus attached the unknown object to a mounting pad so it remains at rest relative to the room.
Hint: *Try to find an analogy to a force you know.*

Fykosaurus should be awarded a Nobel prize for discovering new fundamental interaction.

First, let us think outside the box and solve the problem without integrating or solving differential equations. But we will show such a solution later.

Solving by analogy

What forces in nature do we know? For example, the gravitational or the electrostatic force. However, Kepler's 3rd law applies in those cases, so the period or time of the collision depends on the initial distance. If the force were constant, from some distance r_1 the flight time would be t_1, but from a greater distance r_2, the point mass would approach the object only by the distance $r_2 - r_1$ in the same time. What if the force increased linearly with distance, as is the case with a spring (or harmonic oscillator)? When stretched, the spring starts to oscillate with a period T. Be aware that this period does not depend on the stretching of the spring! The time it takes the spring to shorten to the equilibrium position is independent of the amplitude of the stretch. In our solution, we need the point mass to reach the object at a time that is independent of the initial distance. Congratulations—the harmonic oscillator model works exactly for our problem!

Let r be the instantaneous distance between the point mass and the strange object, and r_0 be the initial distance. It is clear that when we release the point mass from r_0, it has zero initial velocity, while it will have the maximum velocity when $r = 0$. The situation is thus analogous to the shortening of the spring from a stretched position to its equilibrium position. The object on the spring will swing to the other side, stop, move back to the equilibrium position, and finally to its initial position. All four parts take the same amount of time, precisely one-quarter of the period. The motion from the stretched to the equilibrium position therefore takes $T/4$, where T is the period.

The equation for the period of a harmonic oscillator is

$$T = 2\pi\sqrt{\frac{m}{k}}\,,$$

where m is the mass of the object on the spring and k its stiffness, i.e., how much force we need to stretch the spring. In our case we know $t = T/4$ and the mass m, so we need to determine k. We can obtain it as

$$t = \frac{\pi}{2}\sqrt{\frac{m}{k}} \quad \Rightarrow \quad k = \frac{\pi^2 m}{4t^2}\,.$$

For a harmonic oscillator, the dependence of the force on the distance is $F = -kx$. In our case we just rewrite x as r, then substitute for k and get

$$F(\mathbf{r}) = -\frac{\pi^2 m}{4t^2}\mathbf{r}\,.$$

The negative sign indicates mutual attraction. All along, we have been solving the situation in 1D, but the problem is spherically symmetric, so we actually work with vectors.

Solving by calculation

Let us try to support our previous solution by a calculation. However, as it turns out, this calculation will not be completely straightforward, and at one point, we will still have to guess the solution. In the process, however, we will try to outline why all the aforementioned adjustments are natural for a physicist.

We will start by expressing the time it takes for the collision to occur. To do this, we will denote the potential energy of a point mass in the field of an unknown force as $V(r)$. We will write down the law of conservation of energy—it has the form

$$V(r_0) = V(r) + \frac{1}{2}mv^2\,.$$

From here, we can express the velocity as a function of the distance from the object as

$$v(r) = -\sqrt{\frac{2}{m}\left(V(r_0) - V(r)\right)}\,,$$

where we choose a minus sign in front of the square root because while the speed increases, the distance from the object decreases. Note that for this to be true, the potential energy in the direction away from the unknown object must increase.

The time it takes for the point mass to reach the object is then

$$t = \int_{r_0}^{0} \frac{1}{v(r)}\,\mathrm{d}r = -\sqrt{\frac{m}{2}}\int_{r_0}^{0}\frac{1}{\sqrt{V(r_0) - V(r)}}\,\mathrm{d}r = \sqrt{\frac{m}{2}}\int_{0}^{r_0}\frac{1}{\sqrt{V(r_0) - V(r)}}\,\mathrm{d}r\,. \tag{1}$$

We would like to differentiate this expression with respect to r_0 and set it equal to zero—this way, we would prove that the time is constant with respect to r_0. Unfortunately, the situation is not as simple, since we find that this derivative does not exist in general. So let us continue with the modifications.

Let us factor out $\sqrt{V(r_0)}$ to continue

$$t = \sqrt{\frac{m}{2V(r_0)}}\int_{0}^{r_0}\frac{1}{\sqrt{1 - \frac{V(r)}{V(r_0)}}}\,\mathrm{d}r\,,$$

and apply the substitution

$$\frac{V(r)}{V(r_0)} = \sin^2 x\,.$$

Then

$$\frac{\mathrm{d}V(r)}{\mathrm{d}r}\,\mathrm{d}r = 2V(r_0)\sin x\cos x\,\mathrm{d}x\,.$$

Substituting this back into the expression for the time, we get

$$t = \sqrt{\frac{m}{2V(r_0)}} \int_{x_1}^{x_2} \frac{1}{\frac{\mathrm{d}V(r)}{\mathrm{d}r}\cos x} V(r_0) 2 \sin x \cos x \,\mathrm{d}x = \sqrt{2V(r_0)m} \int_{x_1}^{x_2} \frac{1}{\frac{\mathrm{d}V(r)}{\mathrm{d}r}} \sin x \,\mathrm{d}x \,.$$

Before we continue, let us note that x depends on r_0, so again, we cannot just simply differentiate. We also need to clarify the integration limits. If the difference $V(r_0) - V(r)$ is finite, so at $r = 0$ the potential energy does not diverge, we can set $V(0) = 0$. Then $x_1 = 0$. Subsequently, for $r = r_0$, it is true that

$$\frac{V(r)}{V(r_0)} = 1 = \sin^2 x_2 \quad \Rightarrow \quad x_2 = \frac{\pi}{2} \,.$$

We then integrate the expression

$$t = \sqrt{2V(r_0)m} \int_0^{\pi/2} \frac{1}{\frac{\mathrm{d}V(r)}{\mathrm{d}r}} \sin x \,\mathrm{d}x \,.$$

This modification ensures that the integral has constant limits. Now we need its result to be equal to $d/\sqrt{V(r_0)}$ (where d is a constant independent of r_0) for the time to be independent of the initial position. Let us try perhaps the simplest option—let us set

$$\frac{1}{\frac{\mathrm{d}V(r)}{\mathrm{d}r}} \sin x = \frac{c}{\sqrt{V(r_0)}} \,, \tag{2}$$

where $c = 2d/\pi$ is a constant independent of position, which guarantees the equality of units.

For the sine, we substitute from relation $\sqrt{V(r)/V(r_0)} = \sin x$, so we have

$$\frac{1}{\frac{\mathrm{d}V(r)}{\mathrm{d}r}} \sqrt{\frac{V(r)}{V(r_0)}} = \frac{c}{\sqrt{V(r_0)}} \,,$$

which we can adjust to

$$\sqrt{V(r)} = c \frac{\mathrm{d}V(r)}{\mathrm{d}r} \,.$$

This is a simple differential equation, we just need to integrate both sides

$$r = 2c\sqrt{V(r)} + C \,,$$

where $C = 0$, because $V(0) = 0$. We finally get the form

$$V(r) = \frac{r^2}{4c^2} \,.$$

Potential energy is proportional to the square of the distance from the object, which corresponds to a harmonic oscillator!

Next, we also have $r = r_0 \sin x$. We can therefore substitute into the equation for time

$$t = \sqrt{\frac{2r_0^2 m}{4c^2}} \int_0^{\pi/2} \frac{1}{\frac{\mathrm{d}V(r)}{\mathrm{d}r}} \sin x \,\mathrm{d}x =$$
$$= \sqrt{\frac{2r_0^2 m}{4c^2}} \int_0^{\pi/2} \frac{1}{\frac{r}{2c^2}} \sin x \,\mathrm{d}x =$$
$$= \sqrt{\frac{2m}{4c^2}} \int_0^{\pi/2} 2c^2 \,\mathrm{d}x = \sqrt{2mc^2}\,\frac{\pi}{2}\,.$$

If we want to compare this result with the result obtained earlier, we set form of the potential energy equal to the energy of the harmonic oscillator. We get

$$\frac{1}{2}kr^2 = \frac{1}{4c^2}r^2\,,$$

from where we can express c^2 as

$$c^2 = \frac{1}{2k}\,,$$

and plug this into the expression for time

$$t = \frac{\pi}{2}\sqrt{\frac{m}{k}}\,,$$

which is exactly the result we obtained from the intuitive approach. The form of the potential energy of the field is therefore

$$V(r) = \frac{1}{2}\frac{\pi^2}{4}\frac{m}{t^2}r^2\,,$$

and force then acts according to the equation

$$F = -\frac{\mathrm{d}V(r)}{\mathrm{d}r} = -\frac{\pi^2}{4}\frac{m}{t^2}r\,,$$

where the minus sign corresponds to the attractive force.

Mathematically, we thus obtained the expected result; however, in the solution, we made two major assumptions, and we are therefore not guaranteed to have found a unique solution. The assumption that eventually led us to the solution was the relation (2). Although at the time it might have seemed reasonable to assume something like that, we have not mathematically proven that this is the only possibility—the integrand could have been something completely different, while leading to the same result. Before that, we also assumed we could set $V(0) = 0$ without loss of generality. Let us quickly try to look at the case where the potential energy at the object's location diverges.

Let us set $V(0) = -\infty$ and choose $V(\infty) = 0$. In the equation (**??**) we factor out $-V(r)$ and obtain

$$t = \sqrt{\frac{m}{2}} \int_0^{r_0} \sqrt{\frac{-1}{V(r)}} \frac{1}{\sqrt{1 - \frac{V(r_0)}{V(r)}}} \,\mathrm{d}r\,.$$

Once again we try to use a substitution, this time

$$\frac{V(r_0)}{V(r)} = \sin^2 x\,,$$

then

$$-\frac{V(r_0)}{V^2(r)}\frac{\mathrm{d}V(r)}{\mathrm{d}r}\,\mathrm{d}r = 2\sin x\cos x\,\mathrm{d}x\,.$$

We substitute this into the integral and adjust the expression

$$t = -\sqrt{\frac{m}{2}}\int_{x_1}^{x_2} V(r_0)\sqrt{\frac{-1}{V(r)}}\frac{1}{\sin x\frac{\mathrm{d}V(r)}{\mathrm{d}r}}2\,\mathrm{d}x = -\sqrt{\frac{m}{2}}\int_{x_1}^{x_2} V(r_0)\sqrt{\frac{-\sin^2 x}{V(r_0)}}\frac{1}{\sin x\frac{\mathrm{d}V(r)}{\mathrm{d}r}}2\,\mathrm{d}x =$$
$$= -\sqrt{2m}\int_{x_1}^{x_2} V(r_0)\sqrt{\frac{-1}{V(r_0)}}\frac{1}{\frac{\mathrm{d}V(r)}{\mathrm{d}r}}\,\mathrm{d}x = \sqrt{2m}\int_{x_1}^{x_2}\sqrt{-V(r_0)}\frac{1}{\frac{\mathrm{d}V(r)}{\mathrm{d}r}}\,\mathrm{d}x.$$

We integrate from $x_1 = 0$ to $x_2 = \pi/2$. We are solving a similar problem as before. Let us try to set again

$$\sqrt{-V(r_0)}\frac{1}{\frac{\mathrm{d}V(r)}{\mathrm{d}r}} = c\,.$$

By adjusting this we get

$$\sqrt{-V(r_0)} = c\frac{\mathrm{d}V(r)}{\mathrm{d}r}\,.$$

We integrate this and obtain

$$\sqrt{-V(r_0)}r = cV(r) + C\,.$$

However, this equation cannot satisfy the condition $V(0) = -\infty$, so we do not get a new result from this part of the solution with this simple assumption—but this does not mean that no solution exists, only that it cannot be found this easily.

Jaroslav Herman
jardah@fykos.org

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