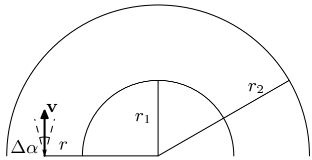
Problem II.5 ... focal point in a cylinder

9 points; průměr 4,33; řešilo 36 studentů

Consider a cylindrical capacitor with internal radius r_1 and external radius r_2. The capacitor is charged so that the voltage between the two electrodes is V. Electrons with a small angle distribution $\Delta\alpha$ are flying out perpendicular to the radius of the cylinder at a distance r ($r_1 < r < r_2$) and at such speed that their distance from the center of the cylinder is approximately constant. Determine the location of the first point in which the electrons focus again. The situation is planar and do not consider the space charge of the electrons.



Jarda heard about different types of analyzers for electron spectroscopy.

We will first express the electric field and the electrostatic potential from the capacitor and then later express the movement of electrons. Gauss's law states that the electric field passing through a surface is proportional to the charge enclosed by that surface. Considering a cylindrical surface with the same axis as the capacitor, we get a zero field for radii smaller than r_1 as the charge inside the surface is zero. For $r_1 > r > r_2$, the cylindrical surface encloses a charge with linear charge density σ, so we easily express the Gauss's law as

$$2\pi r E = \frac{\sigma}{\varepsilon_0}\,,$$

where E is the electric field and ε_0 is the permittivity of vacuum. The electric field is then

$$E = \frac{\sigma}{2\pi\varepsilon_0 r}\,.$$

Outside the cylinder with radius r_2, the enclosed charge is zero, so the electric field is also zero. Now, we can calculate the magnitude of the charge (linear charge density) σ. We do this by calculating the electric potential and determining the potential difference between the inner and outer radii equal to the voltage V.

$$V = \varphi_2 - \varphi_1 = \int_{r_1}^{r_2} \frac{\sigma}{2\pi\varepsilon_0 r}\,\mathrm{d}r$$

$$V = \left[\frac{\sigma}{2\pi\varepsilon_0}\ln r\right]_{r_1}^{r_2} = \frac{\sigma}{2\pi\varepsilon_0}\ln\frac{r_2}{r_1}$$

$$\sigma = \frac{2\pi\varepsilon_0 V}{\ln\frac{r_2}{r_1}}$$

from this equation we can express E as a function of radius as

$$E(r) = \frac{V}{\ln\frac{r_2}{r_1}}\frac{1}{r}\,.$$

Now that we are familiar with the electric field, we can describe the motion of electrons. The only force acting on the charged particles is the electric force, which acts radially (either towards the center or away from it). We will use the polar coordinates to describe the motion. However, since this is a non-inertial reference frame, we must also include the centrifugal force. Thus, in the r and θ coordinates, we get the equation of motion for an electron with a negative elementary charge e

$$m\ddot{r} = mr\dot{\theta}^2 - \frac{eV}{\ln\frac{r_2}{r_1}}\frac{1}{r}\,. \tag{1}$$

Since we have a single equation with two variables, we must introduce another equation to solve the system. This additional equation comes from the law of conservation of angular momentum. The application of this law is valid because the electric force acts radially (toward the center), meaning the net torque is zero. Therefore, we can write

$$L = mr^2\dot{\theta} = \mathrm{const}\,. \tag{2}$$

Thus, we can express angular velocity $\dot{\theta}$ from the equation (2) as

$$\dot{\theta} = \frac{L}{mr^2}$$

and substitute it to the equation (1)

$$m\ddot{r} = mr\left(\frac{L}{mr^2}\right)^2 - \frac{eV}{\ln\frac{r_2}{r_1}}\frac{1}{r}\,,$$

$$\ddot{r} = \frac{L^2}{m^2r^3} - \frac{eV}{m\ln\frac{r_2}{r_1}}\frac{1}{r}\,. \tag{3}$$

Even though we have obtained a differential equation for radius r, we cannot solve it analytically. However, we are interested in a beam with a narrow angular distribution $\Delta\alpha$, which moves along a nearly constant radius trajectory. If the particles are to move along a constant radius, the right-hand side of the equation must be zero. Therefore, the following must be true

$$\frac{L^2}{m^2r^3} = \frac{eV}{m\ln\frac{r_2}{r_1}}\frac{1}{r}\,.$$

From this, we can finally express r_0^2 as

$$\frac{L^2}{m^2r_0^2} = \frac{eV}{m\ln\frac{r_2}{r_1}}\,,$$

$$r_0^2 = \frac{L^2}{m}\frac{\ln\frac{r_2}{r_1}}{eV}\,.$$

Now, we substitute this equilibrium radius corresponding to the angular momentum L into the equation (3)

$$\ddot{r} = \frac{L^2}{m^2}\left(\frac{1}{r^2} - \frac{1}{r_0^2}\right)\frac{1}{r}\,.$$

Next, we must utilize a fun little trick. We cannot solve the equation for arbitrary r, but we know two things – the solution for a stable trajectory at distance r_0 and that the angle $\Delta\alpha$ is small. So, we can expand $r = r_0 + \Delta r$ and linearize the equation.

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}(r_0 + \Delta r) = \frac{L^2}{m^2}\left(\frac{1}{(r_0+\Delta r)^2} - \frac{1}{r_0^2}\right)\frac{1}{(r_0+\Delta r)}$$

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}(\Delta r) = \frac{L^2}{m^2}\left(\frac{r_0^2 - r_0^2 - 2r_0\Delta r - \Delta r^2}{r_0^2(r_0+\Delta r)^2}\right)\frac{1}{(r_0+\Delta r)}$$

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}(\Delta r) = \frac{L^2}{m^2}\left(\frac{-2\Delta r - \frac{\Delta r^2}{r_0}}{r_0(r_0+\Delta r)^2}\right)\frac{1}{(r_0+\Delta r)}\,.$$

We will assume that $\Delta r \ll r_0$ and neglect higher powers of Δr to obtain

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}(\Delta r) = \frac{L^2}{m^2}\frac{-2\Delta r}{r_0^4}\,.$$

Here, we notice the equation of a harmonic oscillator with angular frequency $\omega = \sqrt{2}L/(mr_0^2)$. However, we want to focus on the scenario where the electrons return to the radius r_0, which occurs after half a period corresponding to the time

$$\frac{T}{2} = \frac{\pi}{\omega} = \frac{\pi m r_0^2}{\sqrt{2}L}\,.$$

For small divergent angles of the beam $\Delta\alpha$, we can assume that all electrons have the same angular momentum $L = mr_0^2\dot{\theta}_0$, where $\dot{\theta}_0$ is the initial angular velocity. As we can assume a constant angular velocity in the approximation of small angles, we get the following for the angle θ_1, where all the electrons first converge

$$\theta_1 = \dot{\theta}_0\frac{T}{2} = \dot{\theta}_0\frac{\pi m r_0^2}{\sqrt{2}L} = \dot{\theta}_0\frac{\pi m r_0^2}{\sqrt{2}mr_0^2\dot{\theta}_0} = \frac{\pi}{\sqrt{2}} \doteq 127^\circ\,.$$

This result is utilized, for example, in 127° deflectors, which are used to measure the energy of electrons.

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