Problem II.4 ... a tap and a container

8 points; průměr 4,81; řešilo 108 studentů

We have an empty container of height H. The container has a square base of side length a. A faucet is positioned directly above the container, and at time $t = 0\,\mathrm{s}$, water begins to flow out of the faucet with an initial velocity v_0. Calculate the dependence of the water level in the tank on time t. The volumetric flow rate of water Q is constant. Assume that Q is small enough for the water level in the container to settle instantaneously at a uniform height. However, do not forget about the time it takes the liquid to fall. *Adam wants to tap into hydromechanics.*

The volume of water in the glass is $V = Qt$. The problem, however, is that this is not the volume below the water surface. The actual volume is lower because up until the glass if full, part of the water volume forms the yet-falling stream between the water in a glass and the faucet.

However, the volume of the stream is not directly proportional to its length. The speed of the falling water increases with its distance from the tap, so the cross-section of the stream decreases. If the initial speed of the water at the faucet is v_0 and the cross section of the stream is S_0, then the volumetric flow rate is $Q = S_0 v_0$. The velocity increases with distance d from the tap as $v = \sqrt{2gd + v_0^2}$, the cross section is then $S = Q/v$.

Let the water fall to distance d from the tap for time t_d. Then the volume of current between point d and the tap is equal to Qt_d. The relationship between t_d and d can be found by solving a quadratic equation as

$$t_\mathrm{d} = \frac{\sqrt{v_0^2 + 2gd} - v_0}{g}\,.$$

We arrive at the same result by integrating the volume of water in the stream to a distance d

$$V_d = \int_0^d S\left(d'\right) \mathrm{d}d' = \int_0^d \frac{Q}{\sqrt{2gd' + v_0^2}} \mathrm{d}d' = \frac{Q}{g}\left(\sqrt{2gd + v_0^2} - v_0\right)\,.$$

Consider a time $t_1 > t_0$, where t_0 is the time the water first touched the bottom of the glass. At time t_1, the water level is at height h and the length of the flow is d, and obviously $H = h + d$. Next, let us introduce the area of the base $A = a^2$. The total volume inside the container is $V = Qt$ and it is the sum of of the volume hA under the water level and the volume in the current V_{H-h}, so

$$Qt = Ah + \frac{Q}{g}\left(\sqrt{2g\left(H - h\right) + v_0^2} - v_0\right)\,.$$

We start solving the equation for h

$$\begin{aligned}
\left(gt - \frac{Ahg}{Q} + v_0\right)^2 &= 2g\left(H - h\right) + v_0^2\,,\\
g^2t^2 - 2gt\frac{Ahg}{Q} + 2v_0gt + \frac{A^2h^2g^2}{Q^2} + v_0^2 - 2\frac{Ahgv_0}{Q} &= 2gH - 2gh + v_0^2\,,\\
\frac{A^2h^2g^2}{Q^2} - 2ght\frac{Ag}{Q} - 2gh\frac{Av_0}{Q} + 2gh + g^2t^2 + 2v_0gt - 2gH &= 0\,,\\
\frac{A^2g^2}{Q^2}h^2 - \frac{2g}{Q}\left(Agt + Av_0 - Q\right)h + g^2t^2 + 2g\left(v_0t - H\right) &= 0\,.
\end{aligned}$$

Now we have obtained a quadratic equation with two roots

$$h = \frac{(Agt + Av_0 - Q) \pm \sqrt{(Agt + Av_0 - Q)^2 - 2A^2g\,(v_0t - H) - g^2t^2A^2}}{\frac{A^2g}{Q}}\,,$$

by simplifying the expression under the square root, we get

$$h = \frac{(Agt + Av_0 - Q) \pm \sqrt{A\left(Av_0^2 + \frac{Q^2}{A} - 2gtQ - 2v_0Q + 2gHA\right)}}{\frac{A^2g}{Q}}\,.$$

We still need to select the correct root and comment on this result. The function $h(t)$ is valid for $t > t_0 = (\sqrt{v_0^2 + 2gH} - v_0)/g$, as for any t below said value the level is at zero height and our assumption $H = h + d$ does not hold. Furthermore, it is clear that for the time $t_\mathrm{f} = V/Q = AH/Q$, the glass will be full, so $h = H$ holds. This result should come out from our equation for h, so let's try substituting $t = t_\mathrm{f}$.

$$\begin{aligned} h &= \frac{\left(\frac{A^2gH}{Q} + Av_0 - Q\right) \pm \sqrt{A\left(Av_0^2 + \frac{Q^2}{A} - 2v_0Q\right)}}{\frac{A^2g}{Q}}\,,\\ &= H + \frac{(Av_0 - Q) \pm (Av_0 - Q)}{\frac{A^2g}{Q}}\,. \end{aligned}$$

It is now evident we must choose the negative root so that $h = H$ holds at time $t = t_\mathrm{f}$.

Finally, we can express the time dependence of the level height as

$$f(t) = \begin{cases} 0, & \text{for } t < t_0.\\ \frac{Qt}{A} + \frac{v_0Q}{Ag} - \frac{Q^2}{A^2g} - \frac{Q}{A^2g}\sqrt{A\left(Av_0^2 + \frac{Q^2}{A} - 2gtQ - 2v_0Q + 2gHA\right)}, & \text{for } t_0 < t < t_\mathrm{f}.\\ H, & \text{for } t > t_f. \end{cases}$$

Jaroslav Herman
`jardah@fykos.org`

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