## Problem II.3 ... floating pyramid

5 points; průměr 2,36; řešilo 121 studentů

*Consider a homogeneous pyramid with density $\rho_\mathrm{j} = 250\,\mathrm{kg{\cdot}m^{-3}}$ floating on water with density $\rho = 1\,000\,\mathrm{kg{\cdot}m^{-3}}$. While floating, its axis is vertical. Is the position more stable when the apex of the pyramid is pointing up or down? The height of the pyramid is $h = 20\,\mathrm{cm}$ and the surface area of its base is $S = 49\,\mathrm{cm^2}$.*

*Lego was thinking about a problem where the pyramid oscillates.*

Consider a homogeneous pyramid with a density $\rho_\mathrm{i}$ floating on a water of density $\rho$ (so $\rho_\mathrm{i} < \rho$). Its axis is vertical while floating. The question is whether it reaches an mechanical equilibrium while its apex is pointing up or down.

If we are to compare the stability of two positions, it suffices to compare their potential energy. Since the pyramid displaces the same amount of water in both cases, the water level will be at the same height in both situations. It is reasonable to take the water level as the zero level of potential energy. However, we must consider not only the potential energy of the pyramid itself (otherwise the pyramid would not float but rest on the bottom) but the sum of the potential energy of the water and the pyramid. The problem does not specify how much water is in the container, but this detail is unnecessary, since we only care about the relative potential energy of the two cases, not the exact amount. We've therefore chosen to calculate the difference in potential energy between the system with the pyramid and a case where the pyramid would be replaced with water of the same mass (so the surface would have the same height again).

The volume of the pyramid is $V_\mathrm{i} = 1/3Bh$. Its mass is $m_\mathrm{i} = V_\mathrm{i}\rho_\mathrm{i} = 1/3Bh\rho_\mathrm{i}$. Therefore, the volume of the displaced water is $V_\mathrm{v} = m_\mathrm{i}/\rho = 1/3Bh\rho_\mathrm{i}/\rho$.

We need to find the position of the pyramid's center of mass. It is not difficult to look up said value on the internet, pyramid's center of mass is at 1/4 of its height. This result can, of course, be reached by integrating.

### *Pointing down*

The submerged part of the pyramid will also have a pyramid shape. Due to the similarity of triangles, it will be a smaller version of the original pyramid. Let $k$ be the ratio of similarity. Then the height of the submerged part is $kh$ and the surface of base $k^2B$. The volume of the submerged part is then $V_p = 1/3k^3Bh$. This volume must equal the volume of the displaced water $V_\mathrm{v}$ and so

$$\frac{1}{3}k^3Bh = \frac{1}{3}B\ h\frac{\rho_\mathrm{i}}{\rho}\,,$$

$$k = \left(\frac{\rho_\mathrm{i}}{\rho}\right)^{1/3}\,.$$

Hence the base will be $h - k\,h = h(1 - (\rho_\mathrm{i}/\rho)^{1/3})$ above water. And since the center of mass is at a quarter of the pyramid height lower (as the pyramid is upside down), the center of mass is at height $h - kh - h/4 = h(3/4 - (\rho_\mathrm{i}/\rho)^{1/3})$. Then the potential energy of the pyramid is equal to

$$E_{i1} = m_\mathrm{i}gh\left(\frac{3}{4} - \left(\frac{\rho_\mathrm{i}}{\rho}\right)^{1/3}\right)\,.$$

As for the water, compared to the case in which the water level would have stayed the same height as now, just without any pyramid, we have to remove the water where the pyramid is submerged, which mathematically can be done by “adding” a pyramid with a negative density $-\rho$ and therefore mass $-m_\mathrm{i}$. At the same time, the pyramid's base is at the height $kh$, so the center of mass will be $kh/4$ under the surface, more accurately at the height $-kh/4$. The potential energy of water here is thus

$$E_{v1} = -m_\mathrm{i} g \left( -h \left( \frac{\rho_\mathrm{i}}{\rho} \right)^{1/3} /4 \right) = m_\mathrm{i} g h \left( \frac{\rho_\mathrm{i}}{\rho} \right)^{1/3} /4 \,.$$

The total potential energy in this case relative to the zero level is

$$E_1 = E_{i1} + E_{v1} = \frac{3}{4} m_\mathrm{i} g h \left( 1 - \left( \frac{\rho_\mathrm{i}}{\rho} \right)^{1/3} \right) .$$

### *Pointing up*

In this case, the submerged part of the pyramid will have the shape of a frustum. The part extending above the surface will be a smaller version of the pyramid itself. Let the ratio of similarity here be denoted as $q$. The volume of the submerged part is then

$$V_\mathrm{p} = V_\mathrm{i} - V_\mathrm{above} = \frac{1}{3} Bh - \frac{1}{3} q^3 Bh = \frac{1}{3} (1 - q^3) Bh \,.$$

This volume must be equal to the volume of the displaced water

$$\frac{1}{3}(1 - q^3) Bh = \frac{1}{3} Bh \frac{\rho_\mathrm{i}}{\rho} \,,$$

$$q = \left( 1 - \frac{\rho_\mathrm{i}}{\rho} \right)^{1/3} .$$

Since the pyramid's height above the surface will be $qh$, the pyramid's base will be the remaining $(1 - q)h$ below the surface. The center of mass of the pyramid will be a quarter of the height above the base, that is, at $-(1 - q)h + h/4 = (q - 3/4)h$. The potential energy of the pyramid itself is then equal to

$$E_{i2} = m_\mathrm{i} g h \left( \left( 1 - \frac{\rho_\mathrm{i}}{\rho} \right)^{1/3} - \frac{3}{4} \right) .$$

However, for the water, missing water volume is in the shape of a frustum, making it harder to find the center of mass directly. One option is to integrate directly. Another one would be to solve the problem in two parts: add the whole pyramid as “negative water”, for which we know the center of mass and then add the actual water back to the portion of the pyramid above the waterline, in which we cancel the “negative water”). The potential energy corresponding to the addition of negative water for the entire pyramid is calculated in the same manner as $E_i 2$, we just substitute mass for $-V_\mathrm{i}\rho$:

$$E_{v2-} = -\frac{1}{3} Bh\rho g h \left( \left( 1 - \frac{\rho_\mathrm{i}}{\rho} \right)^{1/3} - \frac{3}{4} \right) .$$

The potential energy corresponding to the addition of water to the part of the pyramid above the surface is obtained similarly, but we substitute the mass with $V_{\text{above}}\rho = 1/3q^3B\ h\rho$ and the center of mass is $qh/4$ above water

$$E_{v2+} = \frac{1}{3}q^3Bh\rho gh\left(1-\frac{\rho_\mathrm{i}}{\rho}\right)^{1/3}/4\,.$$

The potential energy of water is then

$$\begin{aligned}E_{v2} = E_{v2-} + E_{v2+} &= -V_\mathrm{i}\rho gh\left(\left(1-\frac{\rho_\mathrm{i}}{\rho}\right)^{1/3}-\frac{3}{4}\right)+q^3V_\mathrm{i}\rho gh\left(1-\frac{\rho_\mathrm{i}}{\rho}\right)^{1/3}/4\\
&= V_\mathrm{i}\rho gh\left(-\left(1-\frac{\rho_\mathrm{i}}{\rho}\right)^{1/3}+\frac{3}{4}+\left(1-\frac{\rho_\mathrm{i}}{\rho}\right)\left(1-\frac{\rho_\mathrm{i}}{\rho}\right)^{1/3}/4\right)\\
&= m_\mathrm{i}\frac{\rho}{\rho_\mathrm{i}}gh\left(\frac{3}{4}-\left(3+\frac{\rho_\mathrm{i}}{\rho}\right)\left(1-\frac{\rho_\mathrm{i}}{\rho}\right)^{1/3}/4\right)\\
&= m_\mathrm{i}gh\left(\frac{3\rho}{4\rho_\mathrm{i}}-\left(3\frac{\rho}{\rho_\mathrm{i}}+1\right)\left(1-\frac{\rho_\mathrm{i}}{\rho}\right)^{1/3}/4\right)\,.\end{aligned}$$

The total potential energy in this case relative to the zero level is

$$\begin{aligned}E_2 = E_{i2}+E_{v2} &= m_\mathrm{i}gh\left(\left(1-\frac{\rho_\mathrm{i}}{\rho}\right)^{1/3}-\frac{3}{4}\right)+m_\mathrm{i}gh\left(\frac{3\rho}{4\rho_\mathrm{i}}-\left(3\frac{\rho}{\rho_\mathrm{i}}+1\right)\left(1-\frac{\rho_\mathrm{i}}{\rho}\right)^{1/3}/4\right)\\
&= m_\mathrm{i}gh\left(\frac{3}{4}\left(\frac{\rho}{\rho_\mathrm{i}}-1\right)-\left(3\frac{\rho}{\rho_\mathrm{i}}-3\right)\left(1-\frac{\rho_\mathrm{i}}{\rho}\right)^{1/3}/4\right)\\
&= \frac{3}{4}m_\mathrm{i}gh\left(\frac{\rho}{\rho_\mathrm{i}}-1\right)\left(1-\left(1-\frac{\rho_\mathrm{i}}{\rho}\right)^{1/3}\right)\,.\end{aligned}$$

*Comparison*

So the question remains as to which of these energies is greater, i.e., for example, when $E_2$ is greater

$$\begin{aligned}E_1 &< E_2\\
\frac{3}{4}m_\mathrm{i}gh\left(1-\left(\frac{\rho_\mathrm{i}}{\rho}\right)^{1/3}\right) &< \frac{3}{4}m_\mathrm{i}gh\left(\frac{\rho}{\rho_\mathrm{i}}-1\right)\left(1-\left(1-\frac{\rho_\mathrm{i}}{\rho}\right)^{1/3}\right)\\
1-\left(\frac{\rho_\mathrm{i}}{\rho}\right)^{1/3} &< \left(\frac{\rho}{\rho_\mathrm{i}}-1\right)\left(1-\left(1-\frac{\rho_\mathrm{i}}{\rho}\right)^{1/3}\right)\,,\end{aligned}$$

let us denote the ratio $\rho_\mathrm{i}/\rho = p$. Then we get the inequality

$$0 < p^{1/3}-1+\left(\frac{1}{p}-1\right)\left(1-(1-p)^{1/3}\right)\,,$$

while when this inequality is satisfied, $E_2$ is greater, i.e., the apex-up position is less stable. So nothing is stopping us from substituting in the values from the problem statement, namely the density. We can see that the question does not depend on $h$, and $S$. When we substitute for $\rho_\mathrm{i}$ (as in $p = 0.25$), we get $-0.1$ on the right. Inequality is, therefore, not satisfied and $E_2 < E_1$, which implies that the apex-up position is more stable than the apex-down position.

The solution could easily end there, but since this is a model solution, we will discuss the result a bit more. The inequality probably needs to be solved numerically, e.g., let the computer draw a graph of how the right-hand side depends on $p$. WolframAlpha is convenient for this use case, as it will compute its roots. We, however, will use Gnuplot here.

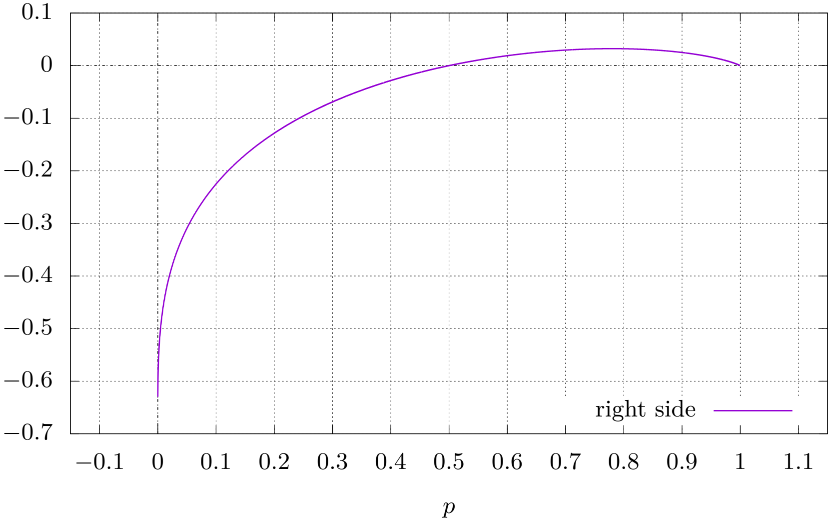


Figure 1: Graph of the right-hand side value as a function of $p$.

We see that for $p = \rho_\mathrm{i}/\rho < 1/2$ the apex-up position is more stable because in such a position the center of mass of the pyramid is lower (imagine, for example, an inflatable pyramid: its density is negligible and it is quite intuitive that it will not "stand on its tip"). Apex-down orientation would be more stable for $p = \rho_\mathrm{i}/\rho > 1/2$ as the water has greater impact here. We even see that for $p = \rho_\mathrm{i}/\rho = 1$ both positions are energetically equally stable, which makes perfect sense, because the density of the pyramid is equal to the density of the water, thus the whole pyramid is submerged, making the potential energy indifferent to orientation.

### *Integrals*

In this section, I will first deduce the position of the pyramid's center of mass by integration, and then the center of mass of the frustum, which we had to solve by a clever trick in a previous section.

Consider a pyramid with height $h$ and base $B$. Its cross-section at height $x$ above the base will be (by similarity, the length decreases linearly) $S(x) = S((h - x)/h)^2 = B(1 - x/h)^2$.

So its mass will be

$$m_\mathrm{i} = \int_0^h \rho_\mathrm{i} B(x)\,\mathrm{d}x = \rho_\mathrm{i} B \int_0^h \left(1 - \frac{x}{h}\right)^2 \mathrm{d}x = \rho_\mathrm{i} B \int_0^h \left(1 - 2\frac{x}{h} + \frac{x^2}{h^2}\right) \mathrm{d}x$$
$$= \rho_\mathrm{i} B \left[x - \frac{x^2}{h} + \frac{x^3}{3h^2}\right]_0^h = \rho_\mathrm{i} B \left(h - h + \frac{h}{3}\right) = \frac{1}{3} Bh\rho_\mathrm{i}\,.$$

We arrived at the same result as we did before. Now we need to calculate height coordinate of the center of mass $h_\mathrm{t}$ by taking the product of mass and height of each point mass and then dividing it the by the mass

$$h_\mathrm{t} m_\mathrm{i} = \int_0^h \rho_\mathrm{i} B(x) x\,\mathrm{d}x = \rho_\mathrm{i} B \int_0^h x \left(1 - \frac{x}{h}\right)^2 \mathrm{d}x = \rho_\mathrm{i} B \int_0^h \left(x - 2\frac{x^2}{h} + \frac{x^3}{h^2}\right) \mathrm{d}x$$
$$= \rho_\mathrm{i} B \left[\frac{x^2}{2} - \frac{2x^3}{3h} + \frac{x^4}{4h^2}\right]_0^h = \rho_\mathrm{i} B \left(\frac{h^2}{2} - \frac{2h^2}{3} + \frac{h^2}{4}\right) = \frac{1}{12} Bh^2 \rho_\mathrm{i}\,,$$

hence $h_\mathrm{t} = (1/12)Bh^2\rho_\mathrm{i}/m_\mathrm{i} = h/4$, as we could find out on the internet.

The coordinates of the center of mass and height of the frustum can be obtained by an analogous procedure with the only difference being not integrating up to the height $h$ but only to the height at which the pyramid is truncated. In our example, the pyramid was cut at a height of $(1-q)h$ above its base. The mass of the displaced water is therefore

$$m_\mathrm{v} = \rho B \int_0^{(1-q)h} \left(1 - 2\frac{x}{h} + \frac{x^2}{h^2}\right) \mathrm{d}x = \rho B \left[x - \frac{x^2}{h} + \frac{x^3}{3h^2}\right]_0^{(1-q)h}$$
$$= \rho B \left((1-q)h - (1-q)^2 h + (1-q)^3 \frac{h}{3}\right)$$
$$= Bh\rho \left((1-q) - (1 - 2q + q^2) + \frac{1}{3}(1 - 3q + 3q^2 - q^3)\right)$$
$$= Bh\rho \frac{1}{3}(1 - q^3) = \frac{1}{3} Bh\rho p = \frac{1}{3} Bh\rho_\mathrm{i} = m_\mathrm{i}\,,$$

which aligns with our previous calculations, as the weight of the displaced water must be equal to the pyramid's weight.

We will now calculate the position of the center of mass in a similar way

$$h_\mathrm{t} m_\mathrm{v} = \rho B \int_0^{(1-q)h} \left(x - 2\frac{x^2}{h} + \frac{x^3}{h^2}\right) \mathrm{d}x = \rho B \left[\frac{x^2}{2} - \frac{2x^3}{3h} + \frac{x^4}{4h^2}\right]_0^{(1-q)h}$$
$$= \rho B \left((1-q)^2 \frac{h^2}{2} - (1-q)^3 \frac{2h^2}{3} + (1-q)^4 \frac{h^2}{4}\right)$$
$$= Bh^2\rho \left(\frac{1}{2}(1 - 2q + q^2) - \frac{2}{3}(1 - 3q + 3q^2 - q^3) + \frac{1}{4}(1 - 4q + 6q^2 - 4q^3 + q^4)\right)$$
$$= Bh^2\rho \left(\frac{1}{12} - \frac{q^3}{3} + \frac{q^4}{4}\right)\,,$$

we see that the height above the base is

$$h_{\rm t} = Bh^2\rho\left(\frac{1}{12}-\frac{q^3}{3}+\frac{q^4}{4}\right)/m_{\rm v} = 3h\frac{1}{p}\left(\frac{1}{12}-\frac{q^3}{3}+\frac{q^4}{4}\right).$$

However, since the base is $(1-q)h$ below the surface, the depth of the center of mass to the surface will also be $(1-q)h$ less than what we just calculated, so

$$\begin{aligned} h_{\rm h} = h_{\rm t} - (1-q)h &= 3h\frac{1}{p}\left(\frac{1}{12}-\frac{q^3}{3}+\frac{q^4}{4}-(1-q)\frac{1-q^3}{3}\right) = 3h\frac{1}{p}\left(-\frac{1}{4}+\frac{q}{3}-\frac{q^4}{12}\right) \\ &= h\left(-\frac{3}{4p}+\frac{q}{p}\left(1-\frac{q^3}{4}\right)\right) = h\left(-\frac{3}{4p}+\frac{q}{4}\left(\frac{4-q^3}{p}\right)\right) = h\left(-\frac{3}{4p}+\frac{q}{4}\left(\frac{3}{p}+1\right)\right). \end{aligned}$$

To get the potential energy of water, we need to multiply the calculation above by $-m_{\rm i}g$ (the expression is negative, because this is the are from which the water is "missing"). If we substitute $q=(1-p)^{1/3}$ and $p=\rho_{\rm i}/\rho$ back into the equation, we get

$$E_{v2} = m_{\rm i}gh\left(\frac{3\rho}{4\rho_{\rm i}} - \left(3\frac{\rho}{\rho_{\rm i}}+1\right)\left(1-\frac{\rho_{\rm i}}{\rho}\right)^{1/3}/4\right),$$

which again, aligns with the results obtained by the different calculation method performed above – as one might expect, it doesn't matter what approach one takes.

Mind that with the parameters included in the problem statement, the stable position is with its apex pointing upward.

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