Problem I.5 ... a tense capacitor

9 points; průměr 4,68; řešilo 79 studentů

Let us consider a plate air capacitor whose plates are connected to the rest of the circuit with springs of stiffness k. In the resting position, the plates are at a distance d and have an area A ($A \gg d^2$). We start charging the capacitor so that the plates begin to attract each other and get closer. Determine the work necessary to charge the capacitor to a charge Q. What is the maximal voltage that we can create between the plates?

Jarda would like to increase the capacity of his head.

From Gauss's law, the electrical intensity near a plate is $E = q/(2\varepsilon_0 A)$. Therefore, the force one plate attracts the other is $F = q^2/(2\varepsilon_0 A)$. This force is compensated by the spring extension, which extends by $\Delta x = F/k$. After charging the plates with a charge of q, the distance between the plates stabilizes at

$$x = d - 2\Delta x = d - \frac{q^2}{\varepsilon_0 k A}\,.$$

We charge the capacitor by moving a charge from one plate to the other. The work done by moving a small charge $\mathrm{d}q$ is

$$\mathrm{d}W = U(q)\,\mathrm{d}q = \frac{q}{C(q)}\,\mathrm{d}q\,,$$

where $U(q)$ is the voltage between the plates as a function of the charge on them and $C(q)$ is the capacitance of the capacitor in dependence on the charge. That is because the capacity changes as the plates move closer. We use a classical equation for capacitance, in which we must consider this change

$$C(q) = \frac{\varepsilon_0 A}{x} = \frac{\varepsilon_0 A}{d - \frac{q^2}{\varepsilon_0 k\ A}}\,.$$

We get the work done as an integral

$$W(Q) = \int_0^Q \mathrm{d}W = \int_0^Q \frac{q\left(d - \frac{q^2}{\varepsilon_0 k\ A}\right)}{\varepsilon_0 A}\,\mathrm{d}q = \frac{Q^2 d}{2\varepsilon_0 A} - \frac{Q^4}{4\varepsilon_0^2 A^2 k}\,.$$

The result can be reached analogically by adding the energy necessary to extend the springs to the energy of the charged capacitor $Q^2/(2C(Q))$. We see that less work needs to be done to charge the capacitor to charge Q in case the plates are able to move than if the plates could not move closer to each other (that corresponds to an infinite stiffness k), even though we had to do some work to extend the springs.

Knowing the dependence of the energy on the charge and the relation between the charge on the plates and the voltage

$$U = \frac{Q\left(d - \frac{Q^2}{\varepsilon_0 k A}\right)}{\varepsilon_0 A}\,,$$

we see that increasing the charge on the plates will lead to the plates moving closer and closer until they touch and the voltage drops to zero. It is therefore obvious, that there exists a maximal voltage, which can be created in the capacitor. We can find it using the derivative of the function $U(Q)$ and finding its zero point

$$\frac{\mathrm{d}U}{\mathrm{d}Q} = \frac{d}{\varepsilon_0 A} - 3\frac{Q^2}{\varepsilon_0^2 A^2 k} \quad\Rightarrow\quad Q_{\max} = \sqrt{\frac{\varepsilon_0 d A k}{3}}\,.$$

Substituting Q_max into the relation for voltage leaves us with the maximal voltage on the capacitor as

$$U_\mathrm{max} = \sqrt{\frac{4k\ d^3}{27\varepsilon_0 A}}\,.$$

By substituting into the equation for the distance we can show that this maximal voltage happens for $x = 2d/3$. It is obvious that at a high enough stiffness of the springs, we can reach arbitrarily high voltage.

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