Problem I.3 ... a preschool with a teacher

5 points; průměr 4,19; řešilo 183 studentů

Jarda is cruising across the town square at a speed of $1.5\,\mathrm{m\cdot s^{-1}}$ *toward his favorite pastry shop. The door to the pastry shop is only* 50 m *ahead of him when he notices a teacher leading a line of preschool children across his path. The teacher is positioned midway between Jarda and the door. The children are following her at a speed of* $0.5\,\mathrm{m\cdot s^{-1}}$ *perpendicular to Jarda's path. Jarda wants to avoid passing through a ten-meter-long line of children to prevent them from accidentally joining him, to avoid inviting them for cake. What is the earliest time Jarda can reach the cake shop if he wants to be walking at constant speed?*

Jarda cannot be stopped on his way to the pastry shop.

First, we will consider whether Jarda could reach the pastry shop by by walking directly along a straight line. He would arrive at the point where his path intersects the line of children at a time $t_0 = d/(2v) \doteq 16.7\,\mathrm{s}$, where $d = 50\,\mathrm{m}$ is his initial distance to the pastry shop, and $v = 1.5\,\mathrm{m\cdot s^{-1}}$ is his speed. Unfortunately, the line of children would only move by $ut_0 \doteq 8.3\,\mathrm{m}$, where $u = 0.5\,\mathrm{m\cdot s^{-1}}$ is their speed, so about 1.7 m of the line is still blocking Jarda's path. He therefore has to modify his route and go around the line.

To get to the pastry shop as soon as possible, it's best to go either past the teacher (who is going first) or the last child. In the previous paragraph, we calculated that by the time he gets to the line, only a few last kids will block his way, so it seems better to go around the kids from behind, not around the teacher at the front. To be sure, we will analyze both versions.

We will consider the angle α_1, under which Jarda will pass around the whole line. We will consider the situation in which Jarda passes the line from the front first. By the time Jarda is passing the line of the children, it would have moved by ut_1 from the initial point perpendicular to Jarda's path. Jarda has to choose his angle α_1 so that the following equation holds

$$\frac{d}{2}\tan\alpha_1 = u\ t_1\,.$$

He also has to reach this point in time t_1. This condition can be expressed as

$$\frac{d}{2}\frac{1}{\cos\alpha_1} = v\ t_1\,.$$

From both equations, we solve for t_1 and obtain

$$\frac{d}{2u}\tan\alpha_1 = \frac{d}{2v}\frac{1}{\cos\alpha_1} \quad\Rightarrow\quad \sin\alpha_1 = \frac{u}{v} \doteq 19.5\,^\circ\,.$$

Jarda will reach the pastry shop in time

$$T_1 = 2t_1 = \frac{d}{u}\frac{\frac{u}{v}}{\sqrt{1-\left(\frac{u}{v}\right)^2}} = \frac{d}{u}\tan\left(\arcsin\left(\frac{u}{v}\right)\right) \doteq 35\,\mathrm{s}\,.$$

We now have to look at the second case where Jarda goes past the last child. He must choose an angle α_2 (which we also consider positive) and reach the last child at time t_2. The (perpendicular) distance from the last child to Jarda's original straight path is $l - ut_2$, where $l = 10\,\mathrm{m}$ is the length of the line of children and also the distance of the child from the perpendicular line. We express time t_2 similarly as before and obtain

$$\frac{1}{u}\left(l - \frac{d}{2}\tan\alpha_2\right) = \frac{d}{2v}\frac{1}{\cos\alpha_2}\,.$$

To simplify, we will divide the equation by l, multiply it by u and introduce auxiliary dimensionless variables $a = ud/(2vl) = 0.833$ and $b = 2d/l = 2.5$, so that

$$a + b\sin\alpha_2 = \cos\alpha_2\,.$$

We square both sides of the equation and use the well-known trigonometric formula $\sin^2 x + \cos^2 x = 1$ to simplify it.

$$(a + b\sin\alpha_2)^2 = 1 - \sin^2\alpha_2 \quad\Rightarrow\quad \left(b^2+1\right)\sin^2\alpha_2 + 2ab\sin\alpha_2 + a^2 - 1 = 0\,,$$

which is a quadratic equation for $\sin\alpha_2$. Its solutions are

$$\sin\alpha_2 = \frac{-ab \pm \sqrt{b^2 - a^2 + 1}}{b^2+1}\,,$$

where the negative root would lead to a negative angle, a solution we are not looking for.

The solution is therefore equal to

$$\sin\alpha_2 = \frac{\sqrt{b^2 - a^2 + 1} - ab}{b^2+1} = \alpha_2 \doteq 3.8^\circ\,,$$

where we substituted numerical values $a \doteq 0.833$ and $b \doteq 2.5$.

The time it takes Jarda to get to the pastry shop is then

$$T_2 = 2t_2 = \frac{1}{v}\frac{d}{\cos\alpha_2} = 33.4\,\mathrm{s}\,.$$

As expected, the time T_2 is lower than the time T_1. For given values, the change in Jarda's direction and his delay are minimal. However, if he noticed the crowd of children later, his delay on the way to the pastry shop would increase.

First, in our solution, we consider whether Jarda could reach the pastry shop by walking directly along a straight line. At the point where his path intersects the line of children, he will arrive after a time $t_0 = d/(2v) \doteq 16.7\,\mathrm{s}$, where $d = 50\,\mathrm{m}$ is his initial distance to the pastry shop and $v = 1.5\,\mathrm{m{\cdot}s^{-1}}$ is his walking speed. However, the line of children will have moved by only $ut_0 \doteq 8.3\,\mathrm{m}$ in that time, where $u = 0.5\,\mathrm{m{\cdot}s^{-1}}$ is the speed of the children, leaving about $1.7\,\mathrm{m}$ of the line still blocking Jarda's path. Jarda must therefore adjust his route and go around the line of children.

To cover the shortest distance and thus reach the pastry shop as quickly as possible, the best strategy is for Jarda to narrowly avoid either the last child or the first teacher. From the previous paragraph, we calculated that by the time Jarda reaches the line, only a few of the last children will still be in his way, suggesting that it is better to go around the children from behind rather than near the teacher. To confirm this, we will calculate both scenarios.

Let us consider the angle α_1 at which Jarda starts to circumvent the entire line. First, we examine the case where he goes around near the teacher. In this case, Jarda and the teacher must meet at a certain point after a time t_1 from the start of his detour. During this time t_1, the line of children moves a distance ut_1 perpendicular to Jarda's original path. Therefore, Jarda must choose an initial angle α_1 such that the following holds:

$$\frac{d}{2}\tan\alpha_1 = u\;t_1\,.$$

At the same time, Jarda must also reach this point in the same time t_1, which is expressed by the condition:

$$\frac{d}{2}\frac{1}{\cos\alpha_1} = vt_1\,.$$

From these two equations, we express t_1 and substitute into their equality:

$$\frac{d}{2u}\tan\alpha_1 = \frac{d}{2v}\frac{1}{\cos\alpha_1} \quad\Rightarrow\quad \sin\alpha_1 = \frac{u}{v} \doteq 19.5\,^\circ\,.$$

Jarda will thus reach the pastry shop in time:

$$T_1 = 2t_1 = \frac{d}{u}\frac{u/v}{\sqrt{1-(u/v)^2}} = \frac{d}{u}\tan\left(\arcsin\left(\frac{u}{v}\right)\right) \doteq 35\,\mathrm{s}\,.$$

Next, we examine the second case, where Jarda goes around the line near the last child. In this case, he must choose an angle α_2 (also considered positive) and reach the child after a time t_2. The distance of the last child from the perpendicular to Jarda's original path is $l - ut_2$, where $l = 10\,\mathrm{m}$ is the length of the line of children and also the distance of the child from the perpendicular at the moment Jarda changes direction. From Jarda's movement, we express the time t_2 as in the previous case, leading to the equation:

$$\frac{1}{u}\left(l - \frac{d}{2}\tan\alpha_2\right) = \frac{d}{2v}\frac{1}{\cos\alpha_2}\,.$$

For clarity, we divide this equation by l, multiply by u, and introduce auxiliary dimensionless variables $a = ud/(2vl) = 0.833$ and $b = 2d/l = 2.5$. After these adjustments, we have:

$$a + b\sin\alpha_2 = \cos\alpha_2\,.$$

Squaring the entire equation and simplifying using the known trigonometric identity $\sin^2 x + \cos^2 x = 1$, we obtain:

$$(a + b\sin\alpha_2)^2 = 1 - \sin^2\alpha_2 \quad\Rightarrow\quad \left(b^2+1\right)\sin^2\alpha_2 + 2ab\sin\alpha_2 + a^2 - 1 = 0\,,$$

which is a quadratic equation for $\sin\alpha_2$. Its solutions are:

$$\sin\alpha_2 = \frac{-ab \pm \sqrt{b^2 - a^2 + 1}}{b^2+1}\,,$$

where the negative sign would lead to a negative angle, which is not our case. Finally, we find:

$$\sin\alpha_2 = \frac{\sqrt{b^2 - a^2 + 1} - ab}{b^2+1} \quad\Rightarrow\quad \alpha_2 \doteq 3.8\,^\circ\,,$$

after substituting $a \doteq 0.833$ and $b \doteq 2.5$.

The time Jarda spends on his way to the pastry shop is then:

$$T_2 = 2t_2 = \frac{1}{v}\frac{d}{\cos\alpha_2} = 33.4\,\mathrm{s}\,.$$

We see that the time T_2 is, as expected, shorter than T_1. For the given numerical values, Jarda's change in direction is minimal, as is his delay. However, if he noticed the line of children later, his delay on the way to the pastry shop would increase.

Jaroslav Herman
jardah@fykos.org

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