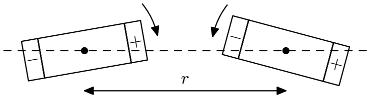
Problem VI.5 ... oscillating magnets

10 points; průměr 5,55; řešilo 22 studentů

Consider two identical dipole magnets, which we fix so that they can rotate in the same plane without friction (their axes of rotation are parallel). If we deflect the magnets slightly out of their equilibrium position, they begin to oscillate. Find the eigenmodes of these oscillations and calculate their frequencies. Discuss what the motion of the magnets will be like for general initial deflection (you don't have to explicitly calculate this case). The magnets have a magnetic moment m, a moment of inertia about the axis of rotation J and the mutual distance between their centers is r.

Jirka stole the problem from Výfuk.



Our task is to study the oscillation of two magnets. When displaced from the equilibrium position, the magnets experience a torque given by $\mathbf{M} = \mathbf{m} \times \mathbf{B}$, which also depends on their displacements from the equilibrium position (as we will see later). That is an example of *coupled oscillations*, which typically lead to a system of linear differential equations. These equations can be elegantly solved using matrix notation, with the eigenmodes corresponding to the eigenvectors of the system matrix. Moreover, we can calculate the frequencies from the eigenvalues. We discussed this method of solving in the Serial of the 34th year *about oscillations and waves*[1]`https://fykos.org/year34/serial/start`. However, understanding why this calculation method works requires advanced knowledge of linear algebra, so we will choose a different approach here.

Before we start with the equations of motion, we will try to guess as many results as possible using physical intuition. We will, of course, verify the accuracy of these guesses with precise mathematical calculations afterward.

Let's consider the magnets in the equilibrium position, where they lie on an axis oriented such that the opposite poles of the magnets are next to each other. When the magnets are slightly displaced, a torque is generated. It is this torque that is returning them to the equilibrium position, and causing them to oscillate in a generally complicated manner. We aim to find the eigenmodes, i.e., such oscillations where the entire system oscillates synchronously with the same frequency. Since the magnets are identical, we can expect that they will oscillate with the same amplitude during such motion. We cannot say much about the phase of the oscillations at first glance, so we will initially look at special cases where the phase is either the same or opposite. In these cases, their absolute displacement (i.e., the absolute value of the displacement) is the same. Thus, the torques exerted by the magnets on each other are of the same magnitude, and the accelerations are the same as well. The situation is perfectly symmetrical, and the motion of the magnets must be identical. Thus, we have found two eigenmodes!

Are these all the modes, or are there any other ones? Each magnet can only rotate in one plane, so we have one degree of freedom per magnet. It can be shown (again, this is a simple result from linear algebra) that the number of modes is equal to the number of degrees of freedom, so our magnets indeed have only two modes – those, we have found earlier. These are illustrated in figures 1 and 2.

[1].

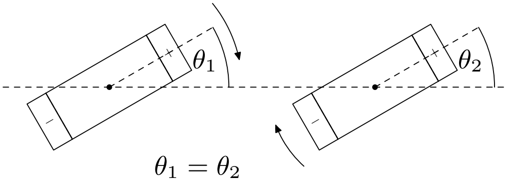


Figure 1: The first eigenmode – the magnets oscillate synchronously such that the displacements from the equilibrium positions of the magnets are the same throughout the motion.

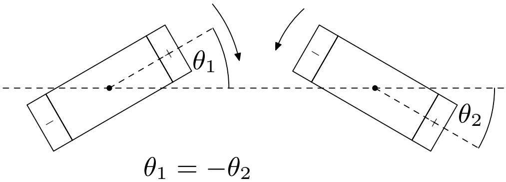


Figure 2: The second eigenmode – the magnets oscillate synchronously such that the displacements from the equilibrium positions of the magnets are opposite throughout the motion.

With the help of simple physics notions, we have thus found candidates for the eigenmodes. Now, let's verify them with precise mathematical calculations. This verification process will not only confirm our candidates but will also determine the corresponding eigenfrequencies.

From the problem statement, we know that the magnets are dipoles. For instance, on Wikipedia[2], we find that the magnetic flux density caused by a magnetic dipole placed at $\mathbf{r} = 0$ at a point with position $\mathbf{r}$ is given by

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi} \left(\frac{3(\mathbf{m} \cdot r)}{r^5} \mathbf{r} - \frac{\mathbf{m}}{r^3} \right) ,$$

where $r = |\mathbf{r}|$ is the distance of the point from the magnet. In our problem, the magnets are at a constant distance r, with only their orientation changing. Let θ_1, θ_2 be the displacements of the magnets from the equilibrium position, as shown in Figures 1 and 2, and let the x-axis be the line on which the centers of the magnets lie. In these coordinates, the magnetic moments of the magnets have components $\mathbf{m}_{1,2} = m\,(\cos\theta_{1,2}, \sin\theta_{1,2})$. Also, the position vector of the second magnet relative to the first one is $\mathbf{r}_1 = (r, 0)$, and in the opposite case, we have $\mathbf{r}_2 = (-r, 0)$. If we displace the first magnet from the equilibrium position by an angle θ_1, the second magnet "feels" the magnetic flux density

$$B_{x1} = \frac{\mu_0}{4\pi} \left(\frac{3mr^2 \cos\theta_1}{r^5} - \frac{m\cos\theta_1}{r^3} \right) = \frac{\mu_0 m}{2\pi r^3} \cos\theta_1 ,$$

$$B_{y1} = \frac{\mu_0}{4\pi} \left(\frac{3mr\cos\theta_1}{r^5} \cdot 0 - \frac{m\sin\theta_1}{r^3} \right) = -\frac{\mu_0 m}{4\pi r^3} \sin\theta_1 .$$

[2] `https://en.wikipedia.org/wiki/Magnetic_dipole`

Similarly, if the second magnet is displaced by θ_2, the flux density at the location of the first magnet is

$$B_{x2} = \frac{\mu_0 m}{2\pi r^3}\cos\theta_2\,,$$
$$B_{y2} = -\frac{\mu_0 m}{4\pi r^3}\sin\theta_2\,.$$

A magnet in an external magnetic field experiences a torque given by $\mathbf{M} = \mathbf{m}\times\mathbf{B}$. In our problem, the torque acting on the second magnet is

$$\mathbf{M}_2 = \mathbf{m}_2\times(\mathbf{B}_{x1}+\mathbf{B}_{y1})\,,,$$

from which

$$M_2 = -mB_{x1}\sin\theta_2 + mB_{y1}\cos\theta_2 = -\frac{\mu_0 m}{4\pi r^3}\left(2\cdot\cos\theta_1\sin\theta_2+\sin\theta_1\cos\theta_2\right)\,,$$

and for the torque acting on the first magnet, we get

$$M_1 = -\frac{\mu_0 m}{4\pi r^3}\left(2\cdot\cos\theta_2\sin\theta_1+\sin\theta_2\cos\theta_1\right)\,.$$

When a body with a moment of inertia J experiences a torque M, the angular acceleration is given by $M = J\ddot{\theta}$. From this, we derive the equations of motion for both magnets while considering only small oscillations. Therefore, we use the approximate relations $\sin x\approx x$ a $\cos x\approx 1$. We obtain the system of linear differential equations

$$\ddot{\theta}_1 = -\frac{\mu_0 m}{4\pi r^3 J}\cdot(2\theta_1+\theta_2)\,,$$
$$\ddot{\theta}_2 = -\frac{\mu_0 m}{4\pi r^3 J}\cdot(\theta_1+2\theta_2)\,.$$

We now want to solve this system. There are many ways, such as using the previously mentioned matrix notation or deriving one of the equations twice and using the substitution method, leading to a single fourth-order equation, etc. We will use a trick that, as we will see later, closely relates to the physical intuition we gained at the beginning. Notice that if we add the equations

$$\ddot{\theta}_1+\ddot{\theta}_2 = -\frac{\mu_0 m}{4\pi r^3 J}\cdot(3\theta_1+3\theta_2)\,,$$

and switch to the new variable $\xi = \theta_1+\theta_2$, using the properties of derivatives, we get

$$\ddot{\xi} = -\frac{3\mu_0 m}{4\pi r^3 J}\xi\,,$$

which is the usual equation for a harmonic oscillator. Luckily, we know that the result of this equation gives us harmonic oscillations with frequency

$$\omega_1 = \sqrt{\frac{3\mu_0 m}{4\pi r^3 J}} = \sqrt{3}\omega_0\,.$$

Similarly, if we subtract the original equations and switch to the variable $\eta = \theta_1-\theta_2$, we get

$$\ddot{\eta} = -\frac{\mu_0 m}{4\pi r^3 J}\eta\,,$$

which corresponds to oscillations with frequency

$$\omega_2 = \sqrt{\frac{\mu_0 m}{4\pi r^3 J}} = \omega_0 \,.$$

So, we have that $\xi = \theta_1 + \theta_2$ oscillates with frequency $\sqrt{3}\omega_0$, and $\eta = \theta_1 - \theta_2$ oscillates with ω_0. But how can we imagine this?

One way is to derive it directly from the equations. The general solution for the harmonic oscillator equation is given by the sum of sin and cos functions oscillating with the corresponding frequency

$$\xi = \theta_1 + \theta_2 = A_1 \sin \omega_1 t + A_2 \cos \omega_1 t \,,$$
$$\eta = \theta_1 - \theta_2 = B_1 \sin \omega_2 t + B_2 \cos \omega_2 t \,,$$

where the constants A_1, A_2, B_1, B_2 are determined by the initial conditions. By adding and subtracting these equations, we obtain separate solutions for the angles θ_1 and θ_2.

$$\theta_1 = \left(A_1 \sin \omega_1 t + A_2 \cos \omega_1 t + B_1 \sin \omega_2 t + B_2 \cos \omega_2 t\right)/2$$
$$\theta_2 = \left(A_1 \sin \omega_1 t + A_2 \cos \omega_1 t - B_1 \sin \omega_2 t - B_2 \cos \omega_2 t\right)/2$$

Let's pause on writing mathematical formulas for now and examine the results. We observe that the motion of the magnets is described by the sum of oscillatory motions with frequencies ω_1 and ω_2. That raises an interesting question: could the magnets oscillate solely with the frequency ω_1, for example? From the equations, we see that mathematically this corresponds to the situation where $B_1 = B_2 = 0$. But can this situation occur at all?

There are several ways to argue this. One is that the initial conditions correspond to four independent constraints on the constants A_1, A_2, B_1, and B_2 (indeed, there are four of them: the initial displacements of both magnets and their initial velocities). These conditions give us a system of four linear equations. We can easily verify if $B_1 = B_2 = 0$ can occur when we substitute this condition into the equations and verify that the equation remains solvable. If we substitute $B_1 = B_2 = 0$ into the equation for $\eta = \theta_1 - \theta_2$, we see that the equation is satisfied when $\theta_1 = \theta_2$ throughout the motion. However, this can be ensured if we set $\theta_1(t=0) = \theta_2(t=0) \neq 0$ and $\dot{\theta}_1(t=0) = \dot{\theta}_2(t=0) = 0$.

Similarly, we find that the motion will contain only the frequency ω_2 if we set $\theta_1(t=0) = -\theta_2(t=0) \neq 0$ and zero initial angular velocities. Thus, we found out that the magnets oscillate synchronously with the frequency $\omega_2 = \omega_0$ if $\theta_1 = -\theta_2$, and conversely, the motion with frequency $\omega_1 = \sqrt{3}\omega_0$ occurs when $\theta_1 = \theta_2$. Note that this exactly corresponds to the eigenmodes we found earlier!

We see that physical considerations can often simplify the solution of complex equations. Instead of solving the equations with lengthy mathematical procedures, we can guess the solution using physical intuition and then verify whether our guess was correct. You can try that for yourself when you substitute $\theta_1 = \theta_2$ and $\theta_1 = -\theta_2$ into our differential equations. The solutions to the equations are indeed harmonic oscillations with the corresponding frequencies. In this procedure, we must remember to check if we have found all the independent modes. We have already argued that our problem has two degrees of freedom, and the equations we derived confirm (in our special case) that two degrees of freedom indeed correspond to only two modes.

What will the general solution look like? From the explicit equations for θ_1 and θ_2, we see that they contain both frequencies. Thus, the general motion will be composed of both modes. How much of each mode is represented depends only on the initial conditions.

The general solution being the sum of modes is a common result due to the linearity of our differential equations. In other words, if any two functions $f(x)$ and $g(x)$ solve a linear differential equation, then the function $h(x) = af(x) + bg(x)$, where a, b are arbitrary constants, also solves it (we leave the proof of this statement to the reader as a simple exercise).

Jiří Kohl
jiri.kohl@fykos.org

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