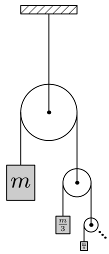
Problem VI.4 ... infinite pulleys

7 points; (chybí statistiky)

Let us have an infinite system of intangible pulleys as shown in the figure, where the mass of each additional weight is one-third of the weight of the previous one. What is the acceleration of the first weight of mass m? *Matěj was looking for the difference between countably and uncountably many pulleys.*



Introducing a substitution

Before we delve into the infinite pulleys, let us derive a rule for simplifying the pulley scheme, which is analogous to the electrical connection of resistors in serial and parallel. Here, we will have just one rule.

Let us first consider the case of a single simple pulley attached to some string on which it exerts a force of F_0. There is one weight on each side of the pulley. Let us denote them by m_1 and m_2. We further denote the acceleration acting on the pulley by a_0. (if the pulley is suspended rigidly, then $a_0 = g$ with respect to the non-inertial system, but this acceleration may be different).

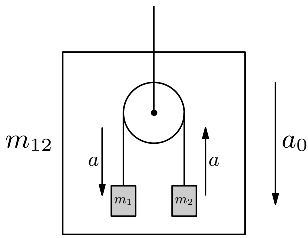


Figure 1: Outline of a simple system.

On a system of weights m_1 and m_2 connected by a string, the forces m_1a_0 and m_2a_0 act in opposite directions. Moreover, according to Newton's second law, both weights must move with the same acceleration a relative to the pulley. That gives us

$$(m_1 + m_2)a = m_1a_0 - m_2a_0 \,,$$

$$a = \frac{m_1 - m_2}{m_1 + m_2} a_0 \,, \tag{1}$$

where we have set a positive direction a for the downward motion of the weight m_1 and m_2 upward (relative to the pulley).

We express the force by which the string is stretched by the acceleration of the first weight

$$F = m_1(a_0 - a).$$

The result is the same for the second weight. We plug in for a

$$F = m_1 \left(1 - \frac{m_1 - m_2}{m_1 + m_2}\right) a_0 = \frac{2m_1m_2}{m_1 + m_2} a_0 \,.$$

The force that stretches the string suspending the pulley is twice as big

$$F_0 = \frac{4m_1m_2}{m_1+m_2}a_0\,. \tag{2}$$

We see that this force is proportional to the acceleration of the pulley. Thus, we can replace the entire pulley with one weight of mass

$$m_{12} = \frac{4m_1m_2}{m_1+m_2}\,, \tag{3}$$

which does not change the net force on the top string.

Replacing with infinity

Let us denote the coefficient of mass decay in the system $k = 1/3$ and solve the whole problem for the general case where k can be any positive number.

Our infinite system of pulleys suspended by one string can also be replaced by a single weight whose effect on the string will be the same. Let us denote the as-yet-unknown mass of such a weight by M.

Now, let us take the first (and the largest) pulley with the first weight. That leaves us with the same infinite system as last time, but all the masses will be k times different. That means we can replace this system with one weight with a mass of kM (see Fig. 2).

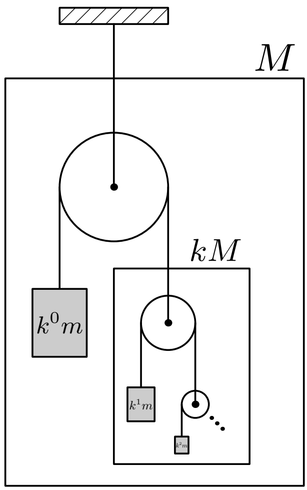


Figure 2: Outline of an infinite system.

So, we have one pulley with weights m and kM. At the same time, we know that when these two weights are replaced by one weight, according to the (3), we must get mass M

$$M = \frac{4mkM}{m+kM}\,.$$

This equation has two solutions for M:

$$M_1 = 0\,,$$
$$M_2 = \frac{(4k-1)m}{k}\,.$$

Solution I (non-physical)

For the first case $M_1 = 0$, the following emerges

$$a_1 = g\,.$$

In this case, all weights fall freely. We can consider it the case where we have a finite (but arbitrarily large) number of pulleys. Thus, there is a final pulley such that no smaller pulley is suspended below it. This string, attached to the last weight in our series, must not be loaded with any counterweight (see (2) for the case of one weight equal to zero $F_0 = 0$). Thus, each additional pulley placed above will also be replaced by a zero mass, meaning no weight will slow any of the higher weights, causing the entire system to free-fall.

This notion is only valid in the idealized case where all strings and pulleys are massless. In that case, it is quite understandable that all the weights will fall to the ground freely because there is always zero mass on the other side of their pulleys.

Solution II (more realistic)

In the second case, we have one pulley with weights m and $kM = (4k-1)m$ on it. Acceleration of the first weight is calculated using the relation (1) (where we plug both masses and $a_0 = g$)

$$a = \frac{m-(4k-1)m}{m+(4k-1)m}g\,,$$

$$a = \frac{1-2k}{2k}g\,.$$

Let's see what this result tells us. If $k = 1/2$, the whole system will be in equilibrium, and the acceleration will be zero. For larger k the acceleration will be upward, and the magnitude will be limitingly close to $-g$. For $k < 1/2$ the first weight will accelerate downward, and the limiting acceleration will approach g.

For our particular case $k = 1/3$ it comes out as

$$a = \frac{1}{2}g\,.$$

Conclusion

In a real-life situation, we will never have ∞ pulleys. In physics, ∞ is often used only as a mathematical tool to describe large quantities of objects or events because it makes calculations much simpler than if we would have to calculate with a finite number of objects. When a physicist says that some quantity X is infinite, he typically means that in all calculations (formally) he uses the limit $\lim_{X\to\infty}$. In most cases, when we have infinite quantities in computation, it does not matter which limit we make first. However, there are examples in mathematics where it indeed depends on the order of limits. As an example

$$\lim_{x\to 1^-}\lim_{n\to\infty} x^n = \lim_{x\to 1^-} 0 = 0\,,$$
$$\lim_{n\to\infty}\lim_{x\to 1^-} x^n = \lim_{n\to\infty} 1^n = 1,.$$

We would run into something similar if we were to solve our pulley problem rigorously. The formal solution would be to solve the example for one pulley, then for two pulleys, and so on. We would need to find a general relationship between the acceleration of the first pulley and the number of pulleys involved in our system. Finally, we would send $n \to \infty$, and we would get the correct result. But the problem is that a system of n pulleys connected in succession has no clear solution because we do not know the tension in the last string. If the tension is zero (there is no counterweight), obviously all n pulleys will free fall. However, if the final string has tension, the pulley system will move in a way that we can accurately predict. Thus, this is a similar problem to the x^n limit order problem. If we add pulleys such that the last string is free, we get the solution I, even if we use ∞ pulleys. On the other hand, if there is any non-zero tension in the last string, the acceleration of the first pulley will gradually converge to the solution. When we add an infinity of pulleys in succession, solution II becomes exact.

Matěj Mezera
m.mezera@fykos.org

FYKOS is organized by students of Faculty of Mathematics and Physics of Charles University. It's part of Media Communications and PR Office and is supported by Institute of Theoretical Physics of CUNI MFF, his employees and The Union of Czech Mathematicians and Physicists. The realization of this project was supported by Ministry of Education, Youth and Sports of the Czech Republic.