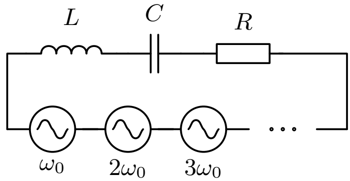
Problem V.5 ... tuning a circuit

9 points; průměr 1,64; řešilo 22 studentů



Consider a series circuit with a resistor of resistance R, a coil, and a capacitor with the capacitance C. AC voltage sources with identical amplitudes U are connected in series with these components. These sources vary in frequency by being multiples of ω_0, where n represents an integer. What frequency, denoted by ω_0, would allow us to find a coil possessing an inductance L, such that voltages with frequencies different from $N\omega_0$ are suppressed by at least 90 % on the resistor? N is a positive natural number known in advance (i.e., the value of ω_0 may depend on it), and we do not want to suppress the voltage with frequency $N\omega_0$ by more than 90 %.

Jarda wanted to have as many different sources in the circuit as possible.

Let's take a look at a simpler situation first. For the voltage across a resistor in a series RLC circuit with an impedance z_n and a voltage source of $U_n = U\sin(n\omega_0 t)$ the following applies

$$U_R = IR = \frac{U}{|z|}R = \frac{R}{\sqrt{R^2 + \left(n\omega_0 L - \frac{1}{n\omega_0 c}\right)^2}}U\,.$$

If we want the voltage on this resistor to be damped by at least 90 %, then it must hold $U_R \leq \alpha U$ for $\alpha = 0.1$.

If we now connect more sources to the circuit as stated in the task, their superposition will give the total voltage. However, the linear behavior of the RLC circuit implies that we can solve for the voltages of different frequencies separately – for each one, we calculate the impedance, and according to the relation above we derive the partial voltage across the resistor.

The condition from the specification then states that, for all $n \neq N$, must hold

$$\frac{R}{\sqrt{R^2 + \left(n\omega_0 L - \frac{1}{n\omega_0 c}\right)^2}} \leq \alpha \tag{1}$$

and at the same time

$$\frac{R}{\sqrt{R^2 + \left(N\omega_0 L - \frac{1}{N\omega_0 c}\right)^2}} > \alpha\,. \tag{2}$$

Let $n \neq N$ and let us solve the inequality (1):

$$\left(\frac{R}{\alpha}\right)^2 \leq R^2 + \left(n\omega_0 L - \frac{1}{n\omega_0 C}\right)^2\,,$$

$$R\sqrt{\frac{1}{\alpha^2} - 1} \leq \left|n\omega_0 L - \frac{1}{n\omega_0 C}\right|\,, \tag{3}$$

from where

$$L \in \mathbb{R}^+ \setminus \left(\frac{1}{n^2\omega_0^2 C} - \frac{A}{n\omega_0}, \frac{1}{n^2\omega_0^2 C} + \frac{A}{n\omega_0}\right)\,,$$

where $A = R\sqrt{1/\alpha^2 - 1}$ a $n \neq N$; for the sake of correctness, let us also mention that by the set $\mathbb{R}^+$ we do not formally mean the set of positive real numbers, but the set of admissible values of coil inductances (the difference is in the unit). By analogy, we also solve the equation (2), where we get

$$L \in \left(\frac{1}{N^2\omega_0^2 C} - \frac{A}{N\omega_0}, \frac{1}{N^2\omega_0^2 C} + \frac{A}{N\omega_0} \right) .$$

If for $n \in \mathbb{N}$ we denote

$$l_n = \frac{1}{n^2\omega_0^2 C} - \frac{A}{n\omega_0} ,$$
$$p_n = \frac{1}{n^2\omega_0^2 C} + \frac{A}{n\omega_0} ,$$

and $J_n = (l_n, p_n)$ interval with these extreme points, we can rewrite the condition from the problem as

$$L \in \left[\mathbb{R}^+ \cap J_N\right] \setminus \bigcup_{\substack{n\in\mathbb{N} \\ n\neq N}} J_n .$$

This can be interpreted to mean that in the system of intervals J_n with the edge points l_n and p_n under study, we find when J_N contains a positive value L that is not also contained in any of the J_n intervals for $n \neq N$.

We will now claim that we can find a satisfying L precisely when $p_{N+1} < l_{N-1}$. To this end, we will make several observations.

1. The sequence p_n consists of positive numbers and decreases monotonically to zero.

2. The sequence l_n also converges to zero, and apparently from some n_0 its terms will always be negative.

3. If ω_0 is such that $l_{N-1} \leq 0$, then necessarily (because of the monotonicity of p_n) will hold

   $$\mathbb{R}^+ \cap (l_{N-1}, p_{N-1}) \supseteq \mathbb{R}^+ \cap (l_N, p_N) ,$$

   therefore the condition of the assignment cannot be fulfilled. Therefore, we will be interested in such frequencies ω_0 where $l_{N-1} > 0$.

4. Let us see when the sequence l_n is monotonic. After a continuous extension of the defining domain[1], we can write

   $$0 > \frac{\mathrm{d}l_n}{\mathrm{d}n} = \frac{1}{n^2\omega_0}\left(A - \frac{2}{n\omega_0}\right) \quad \iff \quad n < \frac{2}{\omega_0 CA} ,$$

   which is specially fulfilled if $l_{n+1} > 0$. For this reason, the sequence l_n is decreasing as long as its values are positive.

5. If $p_{N+1} < l_{N-1}$, then an interval (p_{N+1}, l_{N-1}) has a non-empty intersection with interval J_N. In this intersection we can choose L, this will then be an element of the interval J_N, but since $L > p_{N+1} > p_{N+2} > \dots$, it will not be an element of intervals J_{N+1}, J_{N+2} nor the following. Likewise $0 < L < l_{N-1} < l_{N-2} < \dots < l_1$, is therefore not even an element of intervals J_1 to J_{N-1}.

[1] so that we can derive

6. If, on the other hand, $p_{N+1} \geq l_{N-1}$, we also have $l_{N+1} < l_N < l_{N-1} \leq p_{N+1} < p_N < p_{N-1}$, and so
$$J_N \subseteq J_{N-1} \cap J_{N+1}\,,$$
therefore the condition of the assignment cannot be fulfilled.

Thanks to all of the above, we know that it must hold $p_{N+1} < l_{N-1}$, let us write

$$\frac{1}{(N+1)^2\omega_0^2 C} + \frac{A}{(N+1)\omega_0} < \frac{1}{(N-1)^2\omega_0^2 C} - \frac{A}{(N-1)\omega_0}\,,$$
$$\omega_0 A \left(\frac{1}{N+1} + \frac{1}{N-1}\right) < \frac{1}{C}\left(\frac{1}{(N-1)^2} - \frac{1}{(N+1)^2}\right)\,,$$
$$\omega_0 < \frac{2}{AC}\frac{1}{N^2-1}\,,$$

which is a required condition.

To the extent that the above solution is rather mathematical, we will give some more physical intuition. The RLC circuit has its resonant frequency and the more the source frequency differs from this resonant frequency, the more damped it will be. Therefore, it is sufficient to check only the damping of two adjacent frequencies – because if we damp them sufficiently, voltages with frequencies even further away from the resonant frequency of the circuit will be damped even more (this is exactly the monotonicity we mentioned several times above). The value of L will therefore be chosen so that the resonant frequency $\omega_r = 1/\sqrt{LC}$ was close $N\omega_0$. A more physical approach, where we assume a known shape of the resonance curve, could then look like this: The attenuation of an RLC circuit according to a given frequency is expressed by a resonance curve that has one maximum at the resonant frequency, the width of which is determined by the circuit parameters. The width of the curve at point 90 % attenuation can be expressed from the equation (3). With the use of substitution $A = R\sqrt{1/\alpha^2 - 1}$ we get, for extreme points, considering only positive frequencies

$$\omega_1 = \frac{-AC + \sqrt{A^2C^2 + 4LC}}{2LC}\,,$$
$$\omega_2 = \frac{AC + \sqrt{A^2C^2 + 4LC}}{2LC}\,.$$

The distance between them $\Delta\omega = A/L$ then must be less than $2\omega_0$, so that in an interval with an attenuation of less than 90 % was only the frequency $N\omega_0$ and not the frequency of other sources. From this, we conclude that we are looking for a limit to ω_0, when the adjacent frequency $(N-1)\omega_0$ and $(N+1)\omega_0$ are just the cutoff frequencies ω_1 and ω_2. Target frequency $N\omega_0$ in this case will be exactly in the middle of the interval, i.e. the average ω_1 and ω_2

$$N\omega_0 = \frac{\omega_1 + \omega_2}{2} = \frac{\sqrt{A^2 + C^2 + 4LC}}{2LC} = \sqrt{\frac{A^2}{4L^2} + \frac{1}{LC}}\,.$$

From this relation, we now express the appropriate inductance (choose a positive result)

$$0 = 4L^2CN^2\omega_0^2 - 4L - A^2C\,,$$
$$L = \frac{1 + \sqrt{1 + A^2C^2N^2\omega_0^2}}{2N^2\omega_0^2 C}\,.$$

By setting the inductance to a condition $\Delta\omega < 2\omega_0$ we get

$$\begin{aligned}
2\omega_0 &> \frac{A}{L} = \frac{2N^2\omega_0^2 CA}{1+\sqrt{1+A^2C^2N^2\omega_0^2}}\,,\\
\sqrt{1+A^2C^2N^2\omega_0^2} &> N^2\omega_0 CA - 1\,,\\
1+A^2C^2N^2\omega_0^2 &> N^4\omega_0^2C^2A^2 - 2N^2\omega_0 CA + 1\,,\\
AC(1-N^2)\omega_0 &> -2\,,\\
\omega_0 &< \frac{2}{AC}\frac{1}{N^2-1}\,.
\end{aligned}$$

So we get the same result as in the previous procedure.

Vojtěch David
vojtech.david@fykos.org

Kateřina Rosická
kacka@fykos.org

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