Problem IV.5 ... little Jágr

9 points; průměr 7,56; řešilo 61 studentů

Little Jagr and his friends would like to go out to play ice hockey. However, it has only started freezing recently, so they don't know if the ice on the pond is already thick enough. Calculate how long it takes for a deep pond to freeze sufficiently; if you know that the water temperature is 0 °C at the beginning, the air is kept at a constant −10 °C and the minimum ice thickness for safe skating is 10 cm. Neither the density of the water nor the ice formed changes with depth. The heat transfer between air and ice and water and ice is much faster than heat conduction in ice. You will need to look up the necessary thermal properties of ice.

Aleš's colleague Pepa was reminiscing on his finals at Kepler gymnasium.

We know that water and air exchange heat very quickly. For our purposes, this indicates that the temperature of the ice exposed to the air remains consistently at $T_{vz} = -10\,°C$. Likewise, when the ice is in contact with water, its temperature remains constant at $T_v = 0\,°C$. Consequently, the temperature of the ice fluctuates at various locations, and from the problem's symmetrical nature, it is evident that it is a function of depth y. A heat flux P then flows through the horizontal surface of the ice of area A such that

$$\frac{P}{A} = \lambda \frac{dT}{dy} \,,$$

where $\lambda = 2.2\,W{\cdot}m^{-1}{\cdot}K^{-1}$ is the thermal conductivity of ice. In general, we could expect that the temperature gradient (i.e., the change in temperature depending on depth) is not constant. Then, in a small volume $A\,dy$, we would have, for example, more heat flowing in than flowing out (as indicated by the time-dependent change in the first derivative of the heat capacity relation). This surplus energy would raise the ice's temperature until the temperature gradient reaches equilibrium (which precisely aligns with the principles of the heat conduction equation). The situation in our problem is as follows. We start with a thin layer of ice and keep the boundaries at constant temperatures T_v and T_{vz}. On the other hand, the temperature in the middle of the ice layer changes in an unfamiliar fashion with time. The heat flow passes through the ice and removes heat from the water, changing its state (the ice layer increases in volume). This energy flows through the ice layer, with a portion of it utilized to stabilize the temperature gradient, as discussed earlier. Nonetheless, it is essential to acknowledge that altering the state of a water layer requires significantly more heat than slightly adjusting the temperature of the ice layer. Consequently, the temperature gradient will reach equilibrium much more rapidly than the formation of new ice. Therefore, for the sake of simplicity, we will assume that the relationship between ice temperature and depth y is linear for each ice thickness h. Thus

$$T(y, h) = T_{vz} + \frac{T_v - T_{vz}}{h} \cdot y = T_{vz} + \frac{\Delta T}{h} \cdot y \,,$$

where $\Delta T = T_v - T_{vz}$ denotes the difference between the temperature of the water and the air.

We, now, move on to the calculation of the ice thickness. In a short time dt, the water and the ice with the cross-section A will (due to the conduction) give away the heat of the magnitude

$$dQ = P\,dt = \lambda A \frac{\Delta T}{h}\,dt \,.$$

Part of this heat is used to change the state of water of the mass dm, i.e.

$$dQ_1 = l\,dm = l\rho A\,dh \,,$$

where $\rho = 917\,\mathrm{kg{\cdot}m^{-3}}$ is the density of ice and $l = 334\,\mathrm{kJ{\cdot}kg^{-1}}$ is the specific heat of fusion of the ice. The remaining portion is transformed into cooling the ice itself. Based on our earlier analysis, for the ice temperature to be dependent on both depth y and thickness h, it must satisfy the equation

$$T(y, h + \mathrm{d}h) = T_{\mathrm{vz}} + \frac{\Delta T}{h + \mathrm{d}h} \cdot y\,.$$

Differentiating the temperature relationship with respect to h, we ascertain the rate at which the ice temperature changes with depth y

$$\mathrm{d}T(y) = \frac{\partial T}{\partial h}\,\mathrm{d}h = -\frac{\Delta T}{h^2} \cdot y\,\mathrm{d}h\,.$$

Thus, the heat we must remove from the ice area of the cross-section A and the thickness $\mathrm{d}y$ at the depth of y is given by

$$\mathrm{d}Q_2'(y) = \mathrm{d}mc\,|\mathrm{d}T(y)| = A\rho c\,\mathrm{d}y\frac{\Delta T}{h^2} \cdot y\,\mathrm{d}h\,,$$

where $c = 2.1\,\mathrm{kJ{\cdot}kg^{-1}{\cdot}K^{-1}}$ is the specific heat capacity of ice. Integrating over the entire ice layer then gives the total heat removed from the ice in time $\mathrm{d}t$

$$\mathrm{d}Q_2 = A\rho c\frac{\Delta T}{h^2}\,\mathrm{d}h \int_0^h y\,\mathrm{d}y = \frac{1}{2}A\rho c\Delta T\,\mathrm{d}h\,.$$

Thus, we have obtained that in time $\mathrm{d}t$, the heat conduction removes

$$\mathrm{d}Q = \mathrm{d}Q_1 + \mathrm{d}Q_2 = A\rho\left(l + \frac{1}{2}c\Delta T\right)\mathrm{d}h\,.$$

Here, we can see the significance of our earlier consideration that the temperature of the ice equalizes faster than the ice freezes, since l is approximately 30 times greater than $c\Delta T/2$. We now compare the heat $\mathrm{d}Q$ with the heat power

$$\mathrm{d}Q = P\,\mathrm{d}t = \lambda A\frac{\Delta T}{h}\,\mathrm{d}t\,.$$

From that, we get the differential equation

$$A\rho\left(l + \frac{1}{2}c\Delta T\right)\mathrm{d}h = \lambda A\frac{\Delta T}{h}\,\mathrm{d}t\,,$$

$$\int h\,\mathrm{d}h = \frac{\lambda\Delta T}{\rho\left(l + \frac{1}{2}c\Delta T\right)}\int \mathrm{d}t\,,$$

$$\frac{1}{2}h^2 + C = \frac{\lambda\Delta T}{\rho\left(l + \frac{1}{2}c\Delta T\right)}t\,.$$

From the initial conditions, we find $C = 0$. We substitute for all constants, and we get the following result for the time in which the ice freezes to the required thickness $H = 10\,\mathrm{cm}$

$$t = \frac{\rho\left(l + \frac{1}{2}c\Delta T\right)}{\lambda\Delta T} \cdot \frac{1}{2}H^2 = 19.9\,\mathrm{h} \doteq 20\,\mathrm{h}\,.$$

Jarda Jágr and his friends have to wait for about 20 hours. In real-life scenarios, we typically observe that it takes several days for a pond to freeze adequately. We can attribute this delay to various factors: the initial temperature of the pond water may not be precisely 0 °C, the water in the pond flows, the heat exchange between the ice and the air may not be as fast as we assume, etc.

Deeper reflection on the heat conduction equation

The heat conduction equation states that

$$\frac{\partial T}{\partial t} = \alpha \frac{\partial^2 T}{\partial y^2}\,. \tag{1}$$

In the prior solution approach, we assumed that the temperature within the ice layers evolves over time, indicated by a non-zero term on the left-hand side of the heat conduction equation. However, we also asserted a linear dependency of temperature on depth, resulting in a constant temperature gradient. Consequently, the right-hand side of the heat conduction equation would be zero. This apparent contradiction gives rise to paradoxes in our analysis.

We can easily calculate the ice growth from the temperature gradient at the contact point of ice and water according to the relationship

$$\frac{P}{A} = \lambda \frac{\mathrm{d}T}{\mathrm{d}y}\,.$$

We have assumed that temperature changes linearly with depth within the ice layer, thus ensuring a uniform temperature gradient across all depths

$$\frac{\mathrm{d}T}{\mathrm{d}y} = \frac{\Delta T}{h}\,.$$

The heat removed from the water is

$$\mathrm{d}Q = P\,\mathrm{d}t = \lambda A \frac{\Delta T}{h}\,\mathrm{d}t\,.$$

As a consequence of the heat removal $\mathrm{d}Q$, a portion of the water transforms into a layer of ice with a thickness of $\mathrm{d}h$

$$\mathrm{d}Q = l\,\mathrm{d}m = A\rho l\,\mathrm{d}h\,.$$

Combining the two previous relations gives the differential equation

$$h\,\mathrm{d}h = \frac{\lambda \Delta T}{\rho l}\,\mathrm{d}t\,,$$

whose solution with boundary conditions is

$$\int_0^H h\,\mathrm{d}h = \frac{\lambda \Delta T}{\rho l} \int_0^t \mathrm{d}\tau\,,$$

$$\frac{1}{2}H^2 = \frac{\lambda \Delta T}{\rho l} t\,,$$

$$t = \frac{\rho l}{2\lambda \Delta T} H^2\,,$$

$$t = 19.3\,\mathrm{h} \doteq 19\,\mathrm{h}\,.$$

Based on the assumption of a linear change in temperature with depth, we employed two different methods to reach two different results. Since the term $c\Delta T/2$ is much smaller than the specific heat of fusion of ice l, the numerical results exhibit negligible variance. In your solutions, we have accepted both procedures.

The main flaw in both methods lies in the mistaken assumption of a linear temperature gradient with depth. As demonstrated in equation (1), this assumption precludes any temperature changes over time in this scenario. Instead, it is necessary to solve the partial differential heat conduction equation. The solution of this equation is not at all trivial because the region, in which we are looking for the solution $0 \leq y \leq h$ increases with time $h = h(t)$.

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