## Problem II.P ... height of mountains

10 points; průměr 4,83; řešilo 80 studentů

*Which factors influence the height of mountains on different planets? Make an attempt at a quantitative estimate. You can consider the highest mountains on the Earth, Mars, and other known planets.* *Karel was admiring Olympus Mons.*

Many are familiar with Earth's highest mountain, Mount Everest (8,848 m), but fewer know about Olympus Mons (21,900 m), the tallest peak in the Solar System, situated on Mars. This discrepancy prompts an exploration into why mountains on other planets can reach such staggering heights, and why Earth doesn't boast similarly colossal peaks. This inquiry leads us to a broader question: what factors determine the height of the highest mountains on various planets?

### *Definition of Terms*

We must clarify a few fundamental points before discussing high mountains in more detail. Firstly, the term "mountain" is relatively ambiguous, and many definitions exist, discussing prominence or minimum elevation gain. We will adopt a somewhat non-scientific definition, considering terrain that rises at least 300 m above its surroundings as a mountain. While the definition of a mountain might not be a significant concern, the definition of its height poses more challenges. Earth's reference point is naturally sea level, but no other planet in our solar system has confirmed surface seas. Therefore, several methods are employed to establish a reference point, and we will mention the two most commonly used. The first method involves setting zero height based on pressure. For instance, on Mars, the zero height level is defined where the pressure equals 610 Pa (triple point of water), signifying that liquid water cannot exist below this level. However, this approach relies on the assumption of an atmosphere around the planet, which is not always assured. Therefore, another method involves measuring from the planetary geoid, an equipotential surface of the gravitational field often described as the "shape of the planet if covered only by an ocean".[1] This system is used on the Moon, Venus, Earth, and has also been adopted for Mars. However, for our model, it will be more suitable to consider zero height as the mountain's base, independently of its position relative to the geoid. According to the previous definition, Earth's highest mountain is Mauna Kea, a Hawaiian volcano with most of its height (approximately cca 6000 m) below sea level and the remaining 4 km above.

### *First model*

Let's try to build some initial model. We consider any mountain homogeneous, made of a single material, and the planet a rigid sphere with gravitational acceleration on the surface $g_\mathrm{p}$. If we were to create the tallest object on this planet out of a given material, we would have to guarantee that the ground would not crack under the mountain's weight. The pressure from the mountain's mass would then transmit through the rock, potentially leading to material ejection and, ultimately, the reduction in mountain height. By applying this model, we can approximate the pressure beneath the mountain's summit using the established formula $P_h = h\rho g_p$ (assuming constant gravity acceleration) and compare it with the material's pressure strength, denoted as $\sigma_c$:

$$\sigma_\mathrm{c} > h\rho g_\mathrm{p} \quad \Rightarrow \quad h_\mathrm{max} = \frac{\sigma_c}{g_\mathrm{p}\rho}\,.$$

[1] GPS also uses this system https://en.wikipedia.org/wiki/Geoid

For Earth, let's choose some harder granite with $\sigma_c = 300\,\mathrm{MPa}$ a $\rho = 2\,600\,\mathrm{kg{\cdot}m^{-3}}$[2]. The maximum altitude is $h_{\max} \approx 12\,\mathrm{km}$, comparable to our highest mountain. For the surface of Mars made of basalt[3] we can get altitude of $h_{\max} \approx 30\,\mathrm{km}$.

However, this model has several shortcomings. Firstly, the argument that the material beneath the mountain will crack is not entirely accurate. If the rock is confined, engineering principles tell us that its maximum compressive strength increases because the material has nowhere to escape. The theory that the material, due to applied pressure, would somehow behave like a fluid encounters the same issue – the fluid has nowhere to escape, and its behavior may deviate from conditions observed in a controlled laboratory environment. Additionally, rocks do not follow Pascal's law as closely as we casually assumed, and pressure may not propagate as effectively. A simple counterexample involves using sand as a material. Common experience demonstrates that even gentle pressure on a sandcastle leads to an immediate collapse, highlighting its low compressive strength. However, dunes, composed of the same fragile material, can reach considerable heights without collapsing. Therefore, we should attempt to construct a more sophisticated model that considers the planet's geology and not just material properties.

Nevertheless, this solution holds value as a dimensional analysis. While many parameters influence the height of a mountain, the most crucial ones are gravitational acceleration and material properties (density, strength). All these parameters are included in the equation above, providing appropriate units. Thus, we can assume that the real value will be within a similar order of magnitude.

### *Isostatic equilibrium*

Let's start with Earth, where the concept of isostatic equilibrium prevails. About 250 years ago, people began questioning why, in large mountain ranges like the Himalayas, gravity is approximately the same as everywhere else, even though the mountains should generate a greater gravitational force due to their mass. This observation implies that areal mass should be consistent everywhere, and therefore, pressure should balance somewhere deep beneath the surface. Two explanations emerged to answer how to achieve this. The first method, known as Pratt equilibrium, involves placing more lightweight material beneath the mountain, which does not exert as much gravity and compensates for the added force from the peak. Thus, beneath the summit is a thicker Earth's crust (which has a relatively low density), replacing the denser material of the Earth's mantle that would otherwise be there. The second method is that the mountain itself has a lower density than the rest of the surface, so, in the end, it has the same mass as the surroundings and, therefore, the same gravity. This approach is termed Airy isostatic equilibrium. Over time, researchers have observed that Pratt equilibrium is predominant on continents, while Airy equilibrium prevails on the ocean floors, including oceanic ridges. Given that the tallest peaks are on land, we will adhere to Pratt's equilibrium.

As previously noted, mountains act as a sort of root system to balance gravity, and these roots can only extend as deep as the lithosphere. Otherwise, they would penetrate the asthenosphere, transitioning into a liquid state. We can calculate the potential height of a mountain that would develop roots extending to such depths. If we have a peak with height $h$, the pressure at the surface will be $p_+ = h\rho_{\mathrm{peak}}g$. The pressure drop must offset this change caused by the mountain's root. The initial hydrostatic pressure was $p_0 = d\rho_{\mathrm{mantle}}g$, but if a portion of

[2] `https://www.matweb.com/search/DataSheet.aspx?MatGUID=3d4056a86e79481cb6a80c89caae1d90`
[3] `https://www.matweb.com/search/datasheet.aspx?matguid=9642fc0c67674065923320l627b67b73`

the Earth's mantle is transformed into a mountain root, the pressure will alter to $p_1 = d\rho_{\mathrm{peak}}g$. For equilibrium to occur, the following must hold

$$p_+ = p_0 - p_1 = d(\rho_{\mathrm{mantle}} - \rho_{\mathrm{peak}})g = h\rho_{\mathrm{peak}}g$$

$$h = \frac{\rho_{\mathrm{mantle}} - \rho_{\mathrm{peak}}}{\rho_{\mathrm{peak}}}d\,.$$

If we set the density of the mantle as the density of its main rock (peridotite) $\rho_{\mathrm{mantle}} = 3.4\,\mathrm{g{\cdot}cm^{-3}}$, then $\rho_{\mathrm{peak}} = \rho_{\mathrm{granite}} = 2.7\,\mathrm{g{\cdot}cm^{-3}}$ and $d = t_{\mathrm{lit}} = 100\,\mathrm{km}$, we get

$$h = 26\,\mathrm{km}\,.$$

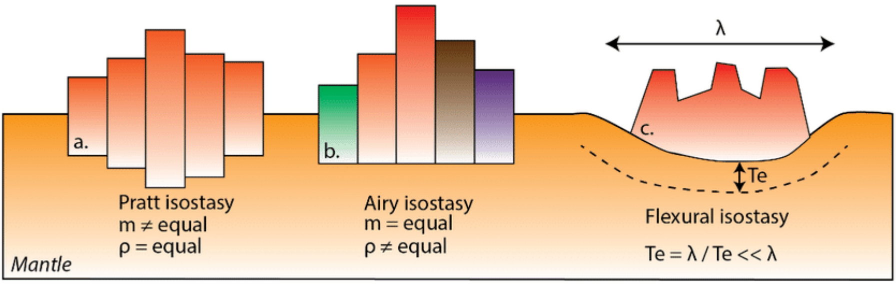


Figure 1: Different types of isostatic equilibrium.
Source: `https://www.researchgate.net/profile/Anouk-Beniest/publication/323184004/figure/fig4/AS:594126588370944@1518662225083/Three-types-of-isostasy-a-Pratt-isostasy-assumes-an-equal-density-and-unequal-mass.png`

This is an upper estimate, as we have not incorporated into the model other phenomena, such as the fact that under such high pressures beneath a mountain, rocks change their structure, forming so-called metamorphic rocks, which generally have a higher density. However, this process is relatively slow and, in the first approximation, does not play a crucial role. Generally, the density of the mountain will change with depth, and Airy's equilibrium will also play its role. For now, we are satisfied with this result.

The question arises as to whether this model can be extrapolated to other planets. Satellite data on Mars reveals significantly higher gravity beneath its mountains, indicating a lack of isostatic equilibrium. Consequently, this model cannot be applied to Mars and, by extension, to other planets. Even on Earth, while this assumption is generally met, it is not universally applicable. A notable exception is Mauna Kea, where gravitational anomalies occur, and neither Pratt's nor Airy's equilibrium is realized. However, an alternative model can be employed, which we will delve into further in the next section.

### *Cracking plate*

If the mountain is not lifted by buoyancy from the Earth's mantle, another mechanism exists to create a force to hold the entire mountain. If we look at Mauna Kea volcano in Hawaii, we find that the bedrock is deforming under the weight of the entire massif. The lithospheric plate here is bending, creating stress inside, which produces a force that tries to push the plate back into its original position. This force pushes against the mountain's weight and causes the whole system to be in equilibrium. This process is well described, for example, in this article, where they employ a similar model to the one we are about to create. The sheer weight of the mountain bends the lithospheric plates, which sink more into the liquid asthenosphere, where the rock melts. At the theoretical limit, we can bend this plate to an angle where the plate breaks, causing the mountain to sink into the depths of the Earth during a significant earthquake.

We can also express this mathematically by borrowing the equations used by engineers to estimate plate bending and plate stress. Imagine a heavy object placed on a plate of a certain thickness and elasticity. If we were to describe the resulting depression accurately, we would have to use stress tensors, which is simply too complicated for our problem. However, envision a scenario with a flexible plate on each side of the mountain as illustrated in figure 2, secured at both ends and attached to the summit. This also converts the problem to 2D. (As an additional possible model, we can also imagine that the plate is connected under the mountain and not divided.) These "beams" bend beneath the summit, and their stiffness, in turn, generates a force that counteracts the weight of the mass.

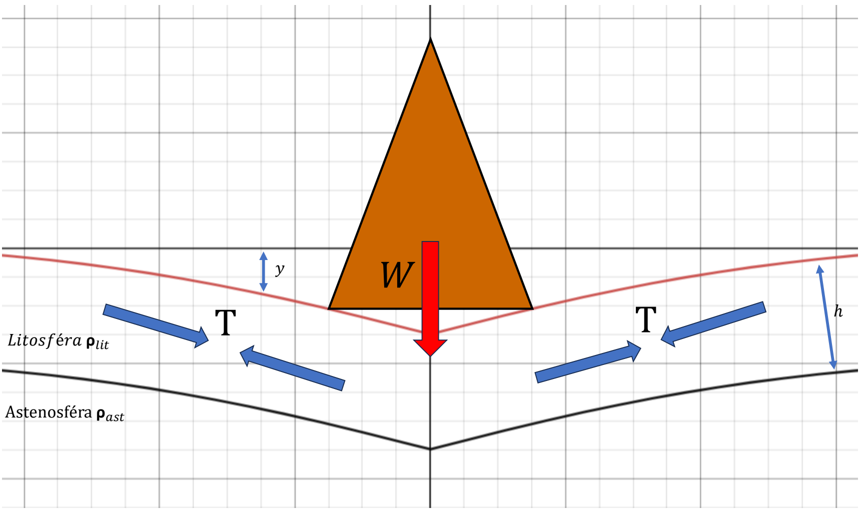


Figure 2: Sketch of a bending litospheric plate.

We can write differential equations for these plates as

$$EI\frac{d^4y}{dx^4} = q\,,$$

where $y(x)$ is a function that tells us how much the terrain has fallen as a function of distance from the center of the mountain, $q$ is the force acting on the plate per unit width and per unit length (i.e., pressure in units Pa), $E$ is Young's modulus of elasticity and $I$ is the so-called second moment of area, which, for a rectangle, is equal to $I = h^3/12$, where $h$ is the height of the plate. This value is related to the width of the plate or the base of the mountain (let's call it $b$). The crucial point for us is value $q$, which we can express using the buoyant force as

$$q = p_{\mathrm{buo}} - p_g = g(\rho_{\mathrm{ast}} - \rho_{\mathrm{lit}})y\,.$$

Note that this term so far only describes the force produced by the immersion of the plate into the asthenosphere; it says nothing about the weight of the mountain itself. It is defined simply by saying the resulting buoyant force must be exactly equal to it.[4] If we add up all this force, we get

$$F_{\mathrm{tot}} = b\int_{-\infty}^{\infty} q\mathrm{d}x = 2b\int_{0}^{\infty} q\mathrm{d}x = 2bg(\rho_{\mathrm{ast}} - \rho_{\mathrm{lit}})\int_{0}^{\infty} y\mathrm{d}x\,,$$

where we focused only on the positive direction $x$ to ensure that the solution is symmetric and calculated only with positive distances $x$, correctly, there should be an absolute value. Let's temporarily put this equation aside and focus on the previous differential equation. If you do not yet have experience solving differential equations, solve them with WolframAlpha, or if you want to see just the solution, you can skip the next section.

With some manipulating

$$EI\frac{d^4y}{dx^4} = g(\rho_{\mathrm{ast}} - \rho_{\mathrm{lit}})y\,,$$
$$\frac{d^4y}{dx^4} - \frac{g(\rho_{\mathrm{ast}} - \rho_{\mathrm{lit}})}{EI}y = 0\,.$$

After declaring of factor $\lambda = \sqrt[4]{g(\rho_{\mathrm{ast}} - \rho_{\mathrm{lit}})/(4EI)}$ we can write the solution of this differential equation as

$$y = (A\sin(\lambda x) + B\cos(\lambda x))\exp(-\lambda x) + (C\sin(\lambda x) + D\cos(\lambda x))\exp(\lambda x)\ .$$

The constants $A, B, C, D$ are determinable from the initial conditions. The terms with a positive exponential must be zero $C = D = 0$ to get a convergent solution. The other two constants will vary depending on which model we are working with. If we consider a model where the plate does not break under the mountain but is continuous, then at point $x = 0$ the plate is horizontal and $dy/\mathrm{d}x = 0$. At the same time, we recall our equation about the mountain's weight, which gave us another piece of information. So, in total

$$0 = \frac{dy}{dx}(y=0) = \lambda\left((-A-B)\sin(\lambda x) + (A-B)\cos(\lambda x)\right)\exp(-\lambda x) \to 0 = A - B\,,$$
$$F_{tot} = W = 2bg(\rho_{\mathrm{ast}} - \rho_{\mathrm{lit}})\int_0^{\infty} y\mathrm{d}x = 2bg(\rho_{\mathrm{ast}} - \rho_{\mathrm{lit}})\frac{A}{\lambda}\,,$$

where $W$ is the weight of the mountain that the plate carries. The constants are therefore $A = B = \lambda W/(2bg(\rho_{\mathrm{ast}} - \rho_{\mathrm{lit}}))$ and the function describing the deflection of the plate comes out as

$$y_{\mathrm{joint}} = \frac{\lambda W}{2bg(\rho_{\mathrm{ast}} - \rho_{\mathrm{lit}})}\left(\cos(\lambda x) + \sin(\lambda x)\right)\exp(-\lambda x)\,.$$

[4] Think about how we could incorporate this information into the equation for $q$ itself.

For the second model, where the plates are broken, it is crucial to consider the distinctions in this scenario. The ends are unattached, implying that there will be no force exerted by the other layers at the ends, resulting in a torque of zero. Upon revisiting the tables, we find that the torque is expressable as

$$M = \frac{EI}{R} \approx EI \frac{d^2 y}{dx^2} = 0,$$

where $R$ is the radius of curvature, which can be approximated by $R \approx \mathrm{d}^2 y/\,\mathrm{d}x^2$. Therefore, the second derivative must be zero. If we use it again the total force condition, the solution is as follows:

$$y_{\text{break}} = \frac{\lambda W}{bg\left(\rho_{\text{ast}} - \rho_{\text{lit}}\right)} \cos\left(\lambda x\right) \exp\left(-\lambda x\right)$$

In both models, the maximum drop is intuitively most significant at the point $x = 0$, and for broken plates, it should be two times larger for the same weight on the plate. We will see later that this factor disappears, and both models predict the same maximum drop. Now, we know how the terrain around the mountain changes with distance, but we still need to solve the stress in the plates. Again, the tables will help us calculate the stress as

$$\sigma = \frac{Et}{2R} = \frac{Et}{2} \frac{d^2 y}{dx^2},$$

where $t$ is the thickness of the lithospheric plate. After fitting for the first model, we get

$$\sigma_{\text{joint}} = \frac{Et}{2} \left| \frac{d^2 y}{dx^2} \right| = Et \frac{\lambda^3 W}{2bg(\rho_{\text{ast}} - \rho_{\text{lit}})} \left| \cos\left(\lambda x\right) - \sin\left(\lambda x\right) \right| \exp\left(-\lambda x\right)$$

and for the second one

$$\sigma_{\text{break}} = \frac{Et}{2} \left| \frac{d^2 y}{dx^2} \right| = Et \frac{\lambda^3 W}{bg(\rho_{\text{ast}} - \rho_{\text{lit}})} \left| \sin\left(\lambda x\right) \right| \exp\left(-\lambda x\right).$$

The highest stress in the first model is at point $x = 0$. For the second one, we have to derive the function and set it equal to zero

$$\frac{d\sigma_{\text{break}}}{dx} = -Et \frac{\lambda^4 W}{g(\rho_{\text{ast}} - \rho_{\text{lit}})} \left(\cos\left(\lambda x\right) - \sin(\lambda x)\right) \exp\left(-\lambda x\right) = 0 \quad \Rightarrow \quad \cos(\lambda x) = \cos(\lambda x)$$

$$x_{\max} = \frac{\pi}{4\lambda}.$$

That is why the maximum stress in the plate is equal to

$$\sigma_{\text{joint}}^{(max)} = Et \frac{\lambda^3 W}{2bg(\rho_{\text{ast}} - \rho_{\text{lit}})},$$

for the model with a broken plate

$$\sigma_{\text{break}}^{(max)} = Et \frac{\lambda^3 W}{\sqrt{2} bg(\rho_{\text{ast}} - \rho_{\text{lit}})} \exp\left(-\frac{\pi}{4}\right).$$

The two models differ only in the numerical prefactor, where for the former, we have $1/2 = 0.5$ and for the second $1/\sqrt{2} \cdot \exp\left(-\pi/4\right) = 0.322$.

Now, let's try to fit some data. For $W$, we will substitute a continuous model plate $W = 1/2 \cdot 1/3 \cdot b^2 H \rho_{\text{lit}} g$, where $H$ is the height of the mountain. Note the coefficient of 1/2, derived from considering that the mountain is supported by two plates perpendicular to each other, parallel to the $x$ or $y$ axis, of the surface. In the second model, we further divide these plates in half because each connected plate is equivalent to two broken ones, yielding $W = 1/4 \times 1/3 \times b^2 H_{\text{lit}} g$. The final values for Earth, Mars and Venus, the three planets where we believe there is a lithosphere with an asthenosphere, are given in table 1.

| | Earth | Mars | Venus |
|---|---|---|---|
| Material of crust | Basalt | Basalt | Basalt |
| Width of lithosphere / km | 100 | 150 | 70 |
| E / Pa | $1.00 \times 10^{11}$ | $1.00 \times 10^{11}$ | $1.00 \times 10^{11}$ |
| The density of lithosphere / kg·kg$^{-3}$ | 2 700 | 2 700 | 2 700 |
| The density of asthenosphere / kg·m$^{-3}$ | 3 300 | 3 550 | 3 400 |
| Tallest mountain | Mauna Kea | Olympus Mons | Skadi Mons |
| Height / m | 10 200 | 21 900 | 10 700 |
| Base / km | 200 | 600 | 700 |
| Gravitational acceleration / m·s$^{-2}$ | 9.8 | 3.7 | 8.9 |
| Compressive strength / MPa | 30 | 30 | 30 |
| Model joint stress / MPa | 370 | 630 | 1 900 |
| Model joint drop / m | 2 800 | 8 000 | 11 600 |
| Model break stress / MPa | 240 | 410 | 1 200 |
| Model break drop / m | 2 800 | 8 000 | 11 600 |

Table 1: Comparison of the properties of the crust and lithosphere of Earth, Mars, and Venus.

We observe that the resulting stress is an order of magnitude higher than the maximum stress of basalt, which does not bode well. According to our model, the mountains should have already collapsed into the asthenosphere and not exist. This represents a notable shortcoming in our model, suggesting we may have overlooked some other crucial phenomena. One explanation could be that the pressure inside the planet is not caused solely by hydrostatic buoyancy but also by some additional thermodynamic processes. However, if that were the case, the function describing the terrain subsidence should differ from reality. Nevertheless, this model has been employed in geology since the last century and has achieved considerable success in locations such as Hawaii.[5] Another possible explanation is that the lithosphere is more robust and has a much higher maximum stress under such extreme conditions. Alternatively, the stress in lithospheric plates may be lower than calculated here, and there might be a mechanism to safely release it. This estimate is, of course, approximate, and we have disregarded numerous factors. For instance, rock density may vary with depth, and several parameters listed in the table for Mars and Venus are more theoretical estimates than data confirmed by actual measurements.

[5] https://www.semanticscholar.org/paper/Gravity-Anomalies-and-Flexure-of-the-Lithosphere-Watts-Cochran/5a583b7da654dacf795ac03df3697549fb9c7294

*Erosion and weathering*

Indeed, the size of a mountain depends not only on how it grows but also on how quickly it diminishes. The most significant ways to achieve this are through erosion and weathering.

Quantifying erosion is quite challenging, and empirical relationships are often necessary, incorporating coefficients for various variables such as precipitation, soil strength, etc. Therefore, here we will attempt to summarize the properties a planet should not have to develop high mountains. First and foremost, erosive phenomena such as wind or rain should not be frequent on a specific planet. Both are common on gas giants, which, despite having a solid core on which mountains could theoretically form, also have strong gas flows in the atmosphere that would likely level any peak in a short time. Additionally, if the atmosphere is very dense, erosion occurs more rapidly. However, whether these factors are crucial for mountain formation is speculative. For example, the highest mountain on Venus (Skadi Mons) is over 6 kilometers high, even though Venus has a very dense and turbulent carbon dioxide atmosphere. This rule is not universal but indicates a trend: we should not expect high mountains on gas giants.

Weathering could similarly impact the size of mountains, with processes generally categorized into chemical and physical (mechanical) processes. Physical weathering may be induced, for instance, by rapid cooling, causing quick contraction of rock that could potentially result in stress and rock fracturing. Such rapid temperature changes occur on planets without atmospheres (like Mercury or even Mars to some extent), where an atmosphere would normally serve as a heat insulator. Whether such an effect influences the maximum height of a mountain is, again, speculative, but it could explain why we do not see very high mountains on Mercury.

The chemical weathering process is somewhat more complex to estimate. On Earth, the primary weathering combination is carbon dioxide with liquid water, but this heavily depends on the type of rock being weathered and the atmosphere. Generally, one might expect this process to be more intense on Venus, where the atmosphere is partially composed of sulfur dioxide. However, we observe that even here, relatively high mountains can form.

The last concept we will mention on this topic is the so-called "glacial buzzsaw." A few years ago, scientists noticed that most mountain peaks are roughly the same distance from the snowline (the altitude where snow persists). It led to the development of the "glacial buzzsaw" concept, which describes erosion at such altitudes.

The main aspects of the "glacial buzzsaw" model involve glacial erosion and the process by which glaciers efficiently weather rocky material and the mountain peaks they traverse. This erosion may stem from the melting and refreezing of glaciers, leading to the physical cracking of rocks and the formation of rock cavities subsequently enlarged through erosive processes. The consequence of this process is the rapid degradation of mountain peaks. Glaciers gradually reduce the overall height of these mountains, maintaining a relatively constant mean elevation. This model explains why some mountains are not higher than expected, given their age, and why some glacial valleys have such a characteristic shape. However, this model is quite controversial; most experiments and measurements attempting to confirm it have not produced conclusive results, and it is not a universally accepted theory.

*Conclusion*

Although we have tried to build a sophisticated model of the Earth's crust, our estimates differ by one order of magnitude from the actual value. A possible explanation is that we have not considered all the possible phenomena and processes occurring in the lithosphere and atmosphere systems. Nevertheless, we have at least an order-of-magnitude estimate using a dimensional analysis in the introduction. The most critical parameters affecting the maximum height of a mountain on a given planet are gravity, the strength of the material, and, in fact, the depth of the asthenosphere.

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