Problem I.S ... measuring the time

10 points; průměr 4,53; řešilo 79 studentů

1. *On long-term average, how long does it take for the March equinox to move by one day when using the Gregorian calendar?*
2. *How much does the period of oscillation of a pendulum with a period of $t = 1\,\mathrm{s}$ change when its temperature changes by $T = 10\,^\circ\mathrm{C}$ if its rod and a much heavier weight are made out of copper? What processes affect the pendulum when the atmospheric pressure or air humidity changes?*
3. *Estimate how long is the shortest "rod" from quartz resonating at a frequency $f = 5\,\mathrm{MHz}$. Consider the density of quartz $\rho = 2.65\,\mathrm{g{\cdot}cm^{-3}}$ and the modulus of elasticity $E \approx 80\,\mathrm{GPa}$ and the compressive oscillations with one static and the other free to move.*
4. *Let's have an isotope ${}^{\mathrm{a}}\mathrm{X}$, that changes with a half-life $T_{1/2}$ to the isotope ${}^{\mathrm{b}}\mathrm{Y}$. At several places in a sample, we measure the relative isotopic abundance of the parent and child nuclides relative to a different isotope of the child element: $[{}^{\mathrm{a}}\mathrm{X}]\,/\,[{}^{\mathrm{c}}\mathrm{Y}]$, $\left[{}^{\mathrm{b}}\mathrm{Y}\right]/\,[{}^{\mathrm{c}}\mathrm{Y}]$. We assume that the relative abundance of the child element does not change in time. How do we determine the age t of the sample? Assume that both isotopes of the element Y are stable and present in original sample and disregard other nuclear transformations.*

1. In the Gregorian calendar, there are 97 leap days over the course of 400 years, as opposed to 100 days in the Julian calendar. The length of the Gregorian year is thus

$$t_{\mathrm{G}} = 365\,\mathrm{days} + \frac{1\,\mathrm{day}\cdot 97}{400} = 365.2425\,\mathrm{days}\,.$$

The duration of the tropical year around the year 2000 is $t_{\mathrm{T}} = 365.24219\,\mathrm{days}$, however it varies slightly in time due to changes in the position of the Earth's axis and the shape of the Earth's orbit. Using this data, we obtain the time it takes for the Gregorian calendar to be erroneous by a day as

$$t = \frac{1\,\mathrm{day}}{T_{\mathrm{T}} - T_{\mathrm{G}}}\cdot 1\,\mathrm{year} \doteq 3\,200\,\mathrm{years}\,.$$

2. For the period of the pendulum we have $T \propto \sqrt{L}$, where L is the length dimension of the pendulum. When the temperature changes, the dimensions of the pendulum change according to the formula for the length expansion

$$L = L_0(1 + \alpha\Delta T)\,,$$

which in our case is the coefficient of longitudinal expansion for copper $\alpha = 16\cdot 10^{-6}\,\mathrm{K^{-1}}$. The new pendulum period will be

$$t = t_0\cdot\sqrt{\frac{L}{L_0}} = \sqrt{1 + \alpha\Delta T} \rightarrow \Delta t \approx \frac{t_0}{2}\alpha\Delta T = 8\cdot 10^{-5}\,\mathrm{s}\,.$$

This change may seem small, but in just one day, it is already cumulatively deviated by $7\,\mathrm{s}$, which is up to three and a half minutes in a month.

The change in air pressure and humidity (with temperature) has a particular effect on the pendulum through the change in air resistance. The movement of the pendulum in a resistive environment leads to a gradual decay of the amplitudes– from time to time, it is necessary to wind the clock's weight, which replaces the lost energy but also changes the period of the clock– as the resistance increases, the period lengthens. The atmospheric pressure also acts in another way – through Archimedes' principle – the buoyant force lightens the weight and thereby changes the effective gravitational acceleration $g_{\mathrm{eff}} = (1 - \rho_{\mathrm{vz}}/\rho_{\mathrm{Cu}})g$ with the variation in the air density ρ_{vz}.

3. If we consider the oscillations in compression along the rod, assuming that one end is fixed and the other one is free, we have for the fundamental mode the formula between the rod's length l and the wavelength λ of the oscillations $l = \lambda/4$ – on one end is a wave node, and on the other is the following mode. The frequency and the wavelength are connected by the speed of propagation of the waves, which is equal to the speed of sound c in the environment as $\lambda f = c$. For a thin rod of density ρ and the modulus of elasticity E, we have the propagation velocity

$$c = \sqrt{\frac{E}{\rho}}\,.$$

By successive substituting, we will get the final equation for the length of "rod"

$$l = \frac{1}{4f}\sqrt{\frac{E}{\rho}} \doteq 0.28\,\mathrm{mm}\,.$$

An interesting alternative was to solve the problem using dimensional analysis since we have three variables (f, E, ρ) and three fundamental quantities contained in units (time, distance, and mass). If we look for the result in the form $l = f^{\alpha} E^{\beta} \rho^{\gamma}$, we will get the following system of equations from the equality of the exponents of the units

$$\begin{aligned}
[\mathrm{s}] &: 0 = -\alpha - 2\beta\,,\\
[\mathrm{m}] &: 1 = -\beta - 3\gamma\,,\\
[\mathrm{kg}] &: 0 = \beta + \gamma\,,
\end{aligned}$$

which give us following solution $\alpha = -1$, $\beta = 1/2$, $\gamma = -1/2$, therefore the equation is

$$l = \frac{A}{f}\sqrt{\frac{E}{\rho}}\,.$$

It is the same expression as the one obtained in the previous calculation, except for the numerical constant A, whose size cannot be determined directly by dimensional analysis. When compared, we see that it has a value of $A = 1/4$.

4. For the representation of the decaying isotope over time, we have

$$\left[{}^{\mathrm{a}}\mathrm{X}\right](t) = \left[{}^{\mathrm{a}}\mathrm{X}\right](t_0)\left(\frac{1}{2}\right)^{\frac{t}{T_{1/2}}}\,.$$

Thus, the amount of the daughter isotope, initially at $\left[{}^{\mathrm{b}}\mathrm{Y}\right](t_0)$, will be gradually increase according to the formula

$$\left[{}^{\mathrm{b}}\mathrm{Y}\right](t) = \left[{}^{\mathrm{b}}\mathrm{Y}\right](t_0) + \left[{}^{\mathrm{a}}\mathrm{X}\right](t_0)\left(1 - \left(\frac{1}{2}\right)^{\frac{t}{T_{1/2}}}\right).$$

Substituting for the unknown original representation of the isotope ${}^{\mathrm{a}}\mathrm{X}$ from the previous equation, we get the equation

$$\left[{}^{\mathrm{b}}\mathrm{Y}\right](t) = \left[{}^{\mathrm{b}}\mathrm{Y}\right](t_0) + \left[{}^{\mathrm{a}}\mathrm{X}\right](t)\left(\left(\frac{1}{2}\right)^{\frac{-t}{T_{1/2}}} - 1\right),$$

which, after dividing by the representation of the isotope ${}^{\mathrm{c}}\mathrm{Y}$ and adjusting the exponential gives us the final equation

$$\frac{\left[{}^{\mathrm{b}}\mathrm{Y}\right]}{\left[{}^{\mathrm{c}}\mathrm{Y}\right]}(t) = \frac{\left[{}^{\mathrm{b}}\mathrm{Y}\right]}{\left[{}^{\mathrm{c}}\mathrm{Y}\right]}(t_0) + \frac{\left[{}^{\mathrm{a}}\mathrm{X}\right]}{\left[{}^{\mathrm{c}}\mathrm{Y}\right]}(t)\left(2^{\frac{t}{T_{1/2}}} - 1\right).$$

Using this equation and the measured abundances of the individual isotopes at time t we can determine ratio of $\frac{\left[{}^{\mathrm{a}}\mathrm{X}\right]}{\left[{}^{\mathrm{c}}\mathrm{Y}\right]}(t)$ and $\frac{\left[{}^{\mathrm{b}}\mathrm{Y}\right]}{\left[{}^{\mathrm{c}}\mathrm{Y}\right]}(t)$ in individual samples with a slightly different chemical composition. It is important to note that since chemically the different isotopes of one same element are (almost) identical, the ration $\frac{\left[{}^{\mathrm{b}}\mathrm{Y}\right]}{\left[{}^{\mathrm{c}}\mathrm{Y}\right]}(t_0)$ is therefore the same in each sample. By applying linear regression to the dependence

$$\frac{\left[{}^{\mathrm{b}}\mathrm{Y}\right]}{\left[{}^{\mathrm{c}}\mathrm{Y}\right]}(t) = A + B \cdot \frac{\left[{}^{\mathrm{a}}\mathrm{X}\right]}{\left[{}^{\mathrm{c}}\mathrm{Y}\right]}(t),$$

we can determine the original ratio $A = \frac{\left[{}^{\mathrm{b}}\mathrm{Y}\right]}{\left[{}^{\mathrm{c}}\mathrm{Y}\right]}(t_0)$ and from the line slope, we can determine the value of the coefficient $B = \left(2^{\frac{t}{T_{1/2}}} - 1\right)$, and from there we get

$$t = T_{1/2} \cdot \log_2(B + 1) = \frac{T_{1/2}}{\ln 2}\ln(B + 1).$$

This method is used, for example, for the isotopes of rubidium or strontium in meteorite dating, which has allowed us to determine the age of the solar system.

Jozef Lipták
`liptak.j@fykos.org`

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