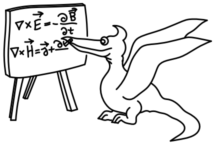

Serial: Length

In the first chapter, we looked at the measurement of time – the most commonly measured quantity in physics in general. In this chapter, we will discuss another quantity that we encounter in everyday life, even as children at school: length. Determining the position of bodies as a function of time makes it possible to describe the motion of bodies (but not the cause of this motion) – an area of physics called kinematics addresses this problem.

Length

Unlike time units, those of length varied throughout history, from one place to another, or in a given place at different times. The first historically documented measurements of length appear together with the first written records – usually describing the area of land and fields, or the lengths or volumes of materials, mainly preserved in records of commercial transactions. The usual measure in antiquity was the cubit (the distance from the elbow to the tip of the middle finger) and, at longer distances, the stadion. These lengths, however, varied from one polity to another (with the monarch usually overseeing their size and observance) and changed over time. By the time of the Roman Empire, the base unit was the foot, divided into twelve parts called unciae, and 5 000 feet formed one mile. Still, the interesting fact remains that, despite the substantial development of geometry in antiquity, measures of length, area, and volume did not form a consistent system, and each had its own system of units.

Due to their widespread use in Europe, the Roman units became the basis for length measurements in this region. The conversion of units, much like the conversion of currencies, was one of the basic tasks that a merchant had to master, since in each country, although often with the same name, the measures were of a different size. For example, in the territory of today's Germany, the mile was different in each federal state. Its size ranged from 1 000 m to 12 000 m. Another problem was the small number of physical standards. For example, the British Standard Yard melted down during the Great Fire of 1834.

Looking for a standard

The development of science and measurement during the Renaissance led to the need for a unified system of units for the scientific community. Such a unit should be determined from natural principles and use decimal multiples. Giovanni Battista Riccioli made the first such proposal in 1645 using the pendulum. In his proposal, one unit of length corresponded to the length of the rod of a pendulum with one second oscillation period. However, as we know from the previous chapter, this definition was soon found unsuitable due to the various influences acting on pendulums, mainly the dependence of the result on the value of the gravitational acceleration, an influence we cannot control by measuring in only one laboratory.

Another proposal was to define the unit of length based on the size of the Earth, which astronomers already used as a unit of length. Gradual advances in geodesy throughout the 17th and 18th centuries showed that the Earth is not perfectly spherical, but is a rotating ellipsoid with a flattening of about 1/300. Thus, during the French Revolution, the length measure *meter* was introduced as follows:

> A meter is 1/10 000 000 of the distance from the North Pole to the equator measured along the Paris meridian.

To implement this definition, two values had to be determined – the flattening of the Earth and a sufficiently long base along the meridian. From measurements in Peru, France, and other locations of the French Empire, the value of the flattening has been recognized as 1/334 (the actual value is about 1/298). The second task consisting of measuring the length of the meridian from Dunkirk to Barcelona took Pierre Méchain and Jean-Baptiste Delambre more than six years to complete. This definition of the meter entered into effect on June 22, 1799, with the deposition of the platinum bar *mètre des Archives* in Paris, and, together with the measures of time and weight, became the foundation of the metric system in December of the same year.[1] By successive remeasurements of the original base distance and its extension from the Shetland Islands to Algeria, the radius and shape of the Earth were refined. Even though the flattening and the base were not determined precisely, the definition was now based on the standard already deposited in Paris. Yet geodetic measurements have not lost their importance. Advances in measurement and processing methods and the linking of geodetic networks between countries[2] have produced geodetic models of the Earth's shape, which have gradually evolved into the World Geodetic System.

As the accuracy of the measurements increased, it was necessary to supplement the conditions at which comparison measurements between an etalon and other standards were performed. The condition on temperature was set in 1889 to be the melting temperature of ice. Another condition of measurement at atmospheric pressure and on a special base that minimizes the tension in the rod itself was added in 1927. In the first half of the 20th century, developments in the precision of the meter physical standard aimed at finding alloys with better thermomechanical properties, such as invar, an alloy of nickel and iron with a length expansion of less than 1 part per million at a temperature change of one kelvin, about ten times less than the original platinum alloy. Charles Édouard Guillaume was awarded the Nobel Prize in 1920 for the discovery of this alloy with outstanding properties.

Measuring workshop

With the industrial development at the turn of the 18th/19th century came a need to manufacture parts of specific dimensions with good precision. It required various new length measurement instrumentation, some of which are still in use today for many different applications.

For the ordinary measurement of distances, tape measures, metersticks, and rulers are used, typically consisting of a 1 mm graduated scale on the straight edge of a rigid body.[3] The measured distance is determined simply by looking at the scale and reading the value from the marks between which the object is located. One should look at the measuring instrument perpendicular to the scale and in good lighting conditions. The measurement's error is usually estimated to be half of the smallest division on the scale. However, this is not entirely accurate. The standard deviation of such a measurement, at a uniform probability density between the lines, can be determined to be $1/\sqrt{3}$ of the smallest division. Furthermore, it is also important to include the accuracy of the measuring instrument given by the accuracy class according to

[1] As early as 1791 was the decision to build a meter according to this definition made, and from April 7, 1795 a prototype with a provisional value of length according to previous measurements was kept in Paris.

[2] One essential contribution was the measurements in the United States, whose use of the meter was a major influential factor in establishing the meter as an international standard.

[3] In the case of a tape measure, it is assumed that the gauge is placed on a flat surface.

the relevant technical standards. For example, for a 10 m long measuring tape of first accuracy class, with a tolerance of 1.1 mm, the total measurement uncertainty is given as

$$\Delta l = \sqrt{(1.1\,\mathrm{mm})^2 + \left(1\,\mathrm{mm}/\sqrt{3}\right)^2} = 1.2\,\mathrm{mm}\,.$$

However, we cannot forget that the measuring device was calibrated under certain standard conditions. If these conditions are not kept during the measurement, it may lead to further systematic errors. For instance, due to length expansion, a 10 m steel measuring tape, calibrated at 20 °C, would be shorter by about 2.5 mm at the freezing point of water.

For more precise measurements of smaller objects, like in mechanical workshops, gauges such as calipers (colloquially called verniers) or micrometers are used. The refinement of measurement is realized both by the use of measuring surfaces – we measure the distance between the two caliper jaws which clamp the measured object – and by the construction of a more complicated scale (e.g., the vernier scale) – allowing the use of smaller divisions. The accuracy of measuring length using calipers is usually 0.05 mm – an order of magnitude better than that of a ruler, and the micrometer is about another order more precise – just 0.005 mm. In both cases, we must subtract the zero point of the scale – the value measured without an object between the measuring claws – from the measured value. Understandably, the measuring surfaces must be kept clean and protected from damage.

For the most accurate measurement commonly encountered in mechanical engineering and for calibration of other instruments, we use gauge blocks (sometimes called Johansson gauges after their inventor) – a set of metal or ceramic blocks with two polished parallel faces at standard distances apart. The choice of distances that make up the set allows you to combine different scales to create a standard of almost any size. The individual gauges are either joined together directly by attaching them in the perpendicular orientation of the measuring walls and rotating them or by applying a layer of oil. Surface tension is to be thanked for the holding of the individual blocks. This assembly is then placed on a sufficiently flat, plane surface together with the object to be measured. The unknown height of the object is determined using a dial indicator, which can measure to an accuracy of up to 0.001 mm, but only within a small range of a few millimeters – which is why the measured quantity is precisely the difference between the standard and the object's size we want to determine. The calibration and remeasurement of standards is done by interferometry – the accuracy of 0.001 mm corresponds to about twice the wavelength of visible light.

Length by light

Apart from everyday life, the mechanical workshop, and geodesy, length measurement also appears in optics. Geometric optics requires surfaces with a sufficiently precise shape when compared to the wavelength of light for maximal image sharpness. The accuracy of the surface execution can be verified by interferometric measurements, by observing interference fringes, rings, or other more complex patterns on the surface. If two waves of electric field amplitude E_0 and phase difference φ meet, the resulting wave will have a light intensity I given as

$$E_0 \mathrm{e}^{i\omega t} + E_0 \mathrm{e}^{i(\omega t + \varphi)} = E_0 \mathrm{e}^{i\omega t}\left(1 + \mathrm{e}^{i\varphi}\right)\,,$$

$$I \propto \frac{E_0^2}{2}\left(1 + \mathrm{e}^{i\varphi}\right)\left(1 + \mathrm{e}^{-i\varphi}\right) = E_0^2\left(1 + \cos\varphi\right)\,,$$

where we used the notation of the wave in one point using complex numbers (the value of the electric intensity is the real part of this expression) and the proportionality of the light intensity to the averaged real part of electric intensity's squared magnitude. Hence, we see that as the phase difference varies from 0 to 2π, the light intensity changes from its maximum value to zero and back to the maximum value. The phase difference is given by the ratio of the width of the medium through which the light travels and the wavelength of the light in this medium or by the ratio of the optical path and the wavelength in vacuum

$$\varphi = \frac{l}{\lambda} = \frac{nl}{\lambda_0}\,.$$

In the previous relation, n is the refractive index of the environment for a given wavelength. Determining the wavelength of a spectral line in meters requires counting the number of stripes that pass through the field of view of an interferometer when the reference surface is displaced by one meter. This is a challenging experiment, particularly when considering the D1 spectral line of sodium, which corresponds to around 1.7 million stripes. For this reason, in the past, the wavelengths of spectral lines have been determined relative to each other, or to a standard wavelength. Albert Abraham Michelson was awarded the Nobel Prize in 1907 for his work in calibrating wavelengths to the standard meter.

The usage of spectroscopy makes it possible to define the meter with a significantly higher accuracy than using the shape of the Earth. For this new definition, choosing a single bright spectral line of an element is necessary. The element must, therefore, only be present as a single isotope in the lamp sample, and the line must not split. The spectral line splits by the interaction between the spin and the orbital momentum of an electron – thus, we have to choose a transition between suitable levels, or the interaction between the electron and the spin of the nucleus – i.e., we have to choose a suitable isotope. In addition, it would be preferable for the element to be gaseous and to glow at the lowest possible temperature so that the line is broadened as little as possible by the thermal motion of the atoms. The isotope chosen in 1960 was the krypton isotope:

> A meter is the length equal to 1 650 763.73 times the wavelength in a vacuum of the radiation emitted by the electron transition from the $2p_{10}$ to the $5d_5$ level of the krypton 86 isotope.

The standard source, thus, became the krypton lamp at the temperature of the triple point of nitrogen $T = 63.151\,\mathrm{K}$.

During the same time period as the new definition of the meter, there was a rapid development in another field – laser optics. A laser is a source not only of monochromatic but also of coherent light. Optical, interferometric, and spectroscopic purposes predestine it as the preferable source. In the 1970s, the helium-neon laser and krypton lamp were remeasured and compared. This comparison led to the realization that by changing the definition to the laser, the accuracy of measurements could increase significantly (up to two orders of magnitude). The choice was thus to fix the value of the speed of light and to measure time instead of length when calibrating standards. To do this, however, measuring the exact value of the source frequencies in the optical domain was necessary but not directly possible. Relative frequency measurements were taken across six orders of magnitude for various sources, ranging from microwave to optical radiation. In 1983, the definition of the meter changed to the following:

> A meter, denoted m, is an SI unit of length defined by fixing the numerical value of the speed of light in vacuum c at 299 792 458 in units of $\mathrm{m{\cdot}s^{-1}}$.

The possibility of measuring length directly from the definition is enabled by laser rangefinders directly measuring the time elapsed between the emission of the laser pulse and the detection of the pulse reflected from the measured object. It is essential to consider the properties of the medium in which we are measuring – we need to know the speed of light in air under the local conditions. We can determine the distance as

$$l = c_\mathrm{g} \Delta t\,, \quad c_\mathrm{g} = \frac{c}{n_\mathrm{g}}\,,$$

$$n_\mathrm{g}(\lambda) = n(\lambda) - \lambda \frac{\mathrm{d}n}{\mathrm{d}\lambda}\,,$$

where it is necessary to use the group velocity of light c_g in the medium given by the group refractive index n_g as opposed to the usual phase refractive index n.[4] On small scales of less than a few meters, interferometry is widely used for accurate measurements – light travels 1 m in about 3.3 ns, so for accuracy to a millimeter, we need to measure time to the accuracy of a picosecond, which is not accessible to an average user. We, therefore, determine the distance by measuring the phase shift φ and the known value of the wavelength of the radiation used.

Astronomical distances

Measuring distances in space has a lot in common with distance measurement in geodesy – historically, geometrical methods were the most accurate for measuring distances. For a long time, the shapes and sizes of planetary orbits were measured in relation to Earth's orbit, which has a semi-axis of one astronomical unit. However, the accurate value of one astronomical unit - which is equivalent to 149,597,870,700 meters - was not known for quite some time. The first somewhat correct values began to appear in the second half of the 17th century from measurements of the parallax[5] of Mars by J. Richer and G. D. Cassini. Later, the value of the solar parallax was refined by the observation of the passings of Venus in front of the Sun in 1761 and 1769, and later in 1874 and 1882. This method had the advantage that instead of measuring the variation of the angular distance when observed from two different locations (the solar parallax has a magnitude of only 8.8 arcseconds), it is based on measuring the time of the transit and followed by some calculations. To determine the size of the astronomical unit from the parallax p, it is further necessary to know the Earth's radius R_Z in the conventional units

$$1\,\mathrm{au} = R_\mathrm{Z} \tan p\,.$$

Today, the most common way to determine distances in the solar system is radar measurement of the time of reflection from a given object, or the time of flight of a signal from a spacecraft (which carries its own precise clock).

Nevertheless, even today, the distances to the stars are measured by the parallax method. F. Bessel measured the first stellar parallax of 61 Cygni in 1838. The basis for measuring stellar parallaxes is the radius of the Earth's orbit, so the star's motion relative to the background over the course of a year is measured. The unit parsec is implemented using the annual parallax –

[4] The pulse of light travels at the group velocity of light c_g as opposed to the wavefront, which travels at the phase velocity c_f. The two velocities are the same in a non-dispersive medium – when the speed of propagation is independent of the wavelength. In fact, a light pulse is the sum of monochromatic waves of different frequencies whose propagation speeds generally differ. In a vacuum, both velocities have the value of the speed of light in vacuum c.

[5] the angle by which the observed object is displaced relative to the distant background when the observer's position changes

it corresponds to the distance at which the star has a parallax of one arcsecond. Thus, we have the relation $d = 1/p$ if we measure the distance d in parsecs and the parallax p in arcseconds. Great advances in the measurements of stellar distances have been achieved by the Hipparchos satellite, capable of measuring distances of stars closer than about 500 pc, and the still active Gaia satellite, with a range about ten times greater. However, not even this distance covers the entire galaxy. For larger distances, we must rely on indirect methods.

In 1908, H.S. Leavitt discovered a relationship between brightness and pulsation period (i.e., the period of variability) by observing Cepheid stars in Magellanic Clouds. By measuring the distances to close Cepheids around the Sun, it is possible to use the observed luminosity to determine the absolute luminosity and thus calibrate the measurement of nearby galaxies using the luminosity of the Cepheids. With these distances, it is then possible to calibrate a so-called standard candle – type Ia supernovae, which are produced as an explosion of a white dwarf on which accretes mass from a companion when the Chandrasekhar limit is exceeded – the maximum possible mass of a stable white dwarf.[6] These allow us to determine the distances in most galaxies that we can distinguish spatially. The main problem is that supernovae are a temporary phenomenon, and to determine the distance of a galaxy using this method, we first need to observe a supernova in it. The distances of galaxies with observed supernovae can then be calibrated to Hubble's law, which says that galaxies are moving away from us at a rate proportional to the distance. By calibration, we mean the determination of the constant of proportionality, known as the Hubble constant. The rate of receding is determined spectroscopically by measuring the Doppler shift of spectral lines. In addition to these methods, there are various other techniques suitable for determining other types of objects or at other distances.

Small distances

So far, we have measured length either by comparison with a physical "ruler" or by using light. But what if the dimensions of the object we are measuring are comparable to the wavelength of visible light, or even smaller? One option is to use radiation with a smaller wavelength, such as ultraviolet or roentgen radiation, and a microscope or interferometer with a camera adapted to this radiation. In recent decades, there has been significant progress in the development of short-wavelength lasers.

If we are interested in distances in a regular structure, such as crystals, we do not need to directly measure the distance. We can look at the crystal as a diffraction grating, where for the distance of the atomic planes d and the wavelength of light λ, we have a relation for the maximum intensity of the reflected light

$$n\lambda = 2d\sin\theta\,,$$

where n is an integer and θ is the angle between the atomic plane of the crystal and the incident or reflected beam. Suppose a beam of roentgen radiation is shone on a powdered sample of crystalline material. In that case, after the radiation passes through the sample, a set of coaxial light cones, whose deviations from the straight line direction are 2θ, will be formed. It is the technique of powder Roentgen diffractometry that measures these angles and the corresponding intensities of the maxima and thus allows us to determine the atomic

[6] The luminosity of such a supernova is up to $3.6 \cdot 10^{9}$ of the solar luminosity. We can see them to distances a thousand times greater when compared to the Cepheids.

distances and the composition of crystalline substances. The structure of biomolecules such as DNA has also been determined using diffraction methods.

Another possibility is to abandon light and use electrons. According to de Broglie's principle, a beam of electrons with momentum p behaves like a light of wavelength

$$\lambda = \frac{h}{p}\,,$$

where h is Planck's constant. Electron microscopes exploit this by using a beam of accelerated electrons focused by magnetic lenses and directed onto a small area of the observed object. With today's technology, scanning electron microscopes are able to achieve resolutions down to $0.5\,\text{Å} = 0.000\,000\,05\,\mathrm{mm}$. The incident electron beam interacts with the inspected sample – afterward, we can observe reflected electrons, secondary low-energy electrons, and Roentgen emission. Those allow us to determine the shape, position, and chemical composition of the sample. In the case of a transmission electron microscope, we do not illuminate the sample point by point, but all at once with a uniform collimated beam, and observe the transmission through the sample. However, sample preparation is principal – a high vacuum is maintained inside the electron microscope, so the sample cannot contain, for example, water. Furthermore, bombarding with electrons charges the measured object, so charge dissipation is crucial – this is why we, for example, have to dry or metalize observed insects.

However, it is necessary to calibrate the scales of the electron microscopes. For this purpose, the interlayer distance d_{220} in pure crystalline silicon was measured with high precision $d_{220} = 192.015571(3) \cdot 10^{-12}\,\mathrm{m}$. Other advanced imaging techniques, such as the atomic force microscope and the scanning tunneling microscope, also use the accuracy of the previous result to their advantage. These techniques measure distance indirectly by detecting the force exerted between the atoms of the sample or the electric current of electrons quantum tunneling between the probe tip and the surface of the measured object. The atomic force microscope moves across the surface and tracks the deflection of an arm with a detection tip, while the scanning tunneling microscope uses vacuum tunneling to measure the current. These modern techniques make it possible to measure distances on a scale even smaller than the dimensions of the atoms themselves.

Derived quantities

The length measurement is also related to other quantities, such as area. We use geometric relationships between the latter and other measured parameters for simple shapes. When calculating the area of planar figures, we typically divide the shape into triangles and sum their areas. However, one must be mindful here. The more triangles we use, the more similar the determined area will be to the original shape, but keep in mind that by using smaller shapes, we are also committing a higher relative error in the measurement. We can apply a similar procedure for non-planar areas as well. In this case, we choose points on the given surface, which we connect into triangles outside of the surface itself. The more points we use, the better will the measured area resemble the actual one. However, here, even more than in the planar case, the position of the individual points is an essential part of the measurement itself. If the shape to be measured can be described by an algebraic equation, it is possible to determine its area by using mathematical analysis – the area integral – and by measuring a few basic lengths describing the dimensions and shape of the body. The basic unit of area is m^2, but we often encounter its multiples – acre $1\,\mathrm{a} = 100\,\mathrm{m}^2$, hectare $1\,\mathrm{ha} = 10\,000\,\mathrm{m}^2$, or the square kilometer $1\,\mathrm{km}^2 = 10 \cdot 10^6\,\mathrm{m}^2$.

The methods are similar in the case of measuring volumes – we can either use the known relations for regular solids or divide the solid into imaginary tetrahedra and proceed the same way as in the previous case. However, there are two other methods – to determine the volume by measuring the mass if we know the density of the body,[7] or by using fluids. Fluids, i.e., gases and liquids, adapt their shape to their surroundings. Thus, we can determine the volume of a fluid using a graduated cylinder by reading the height of the fluid level. We can even measure the volume of other bodies by immersing them in a liquid and noting the volume change. It is crucial to ensure that the fluid and the object do not react, that the object is fully submerged, and that no air bubbles are present due to surface tension.

Using units of length and time, we can describe the time dependence of position and its changes – velocity and acceleration. Velocity is a vector quantity given as the variation of the position vector over a given infinitesimal interval of time. Thus, the velocity vector has a direction given by the direction of the motion in addition to its magnitude. Apart from this so-called instantaneous speed, we also distinguish the average speed – the ratio of distance/path traveled during a given time interval. The base unit of velocity is m/s. Nevertheless, we often encounter $3.6\,\mathrm{km/h} = 1\,\mathrm{m/s}$. As you will see when solving the serial problem, many other units are used in practice. When stating a velocity, it is always important to indicate the reference frame with respect to which the motion is being observed unless it is obvious from convention or context.

Acceleration is also a vector quantity defined as the variation of the velocity vector over an infinitely small interval of time. Thus, a body can have a non-zero acceleration even if it is moving with a velocity of constant magnitude (but of changing direction). The base unit of acceleration is $\mathrm{m/s}^2$, but acceleration is also sometimes given in multiples of the gravitational acceleration $g_0 = 9.80665\,\mathrm{m/s}^2$, or in the case of planetary sciences in the units of galileo $1\,\mathrm{Gal} = 1\,\mathrm{cm/s}^2$. The value of the acceleration is determined either by calculation from the known dependence of the body's position on time or by using an accelerometer. Various types of accelerometers measure acceleration in different ways. Pendulum accelerometers measure acceleration using the period of a pendulum, while spring accelerometers determine acceleration by calculating the inertial force acting on a body on a spring. Gyroscopic accelerometers determine acceleration from the precession velocity of a flywheel, and other types of accelerometers are also available. However, in all cases, they only measure the inertial acceleration of the body (due to the equivalence principle, we cannot distinguish acceleration from gravity), and we usually need three accelerometers to determine the acceleration vector.

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[7] We will discuss mass and density in the next chapter.