23. ročník, úloha II . 3 ... *barrel-organ* *!!! chybí statistiky !!!*

You can buy a roller with small bumps on its surface. The bumps hit an edge of a steel wafer which is divided into several parts of different length. All the tones of an interval (say C major) are played in the song recorded on the roller. Can you determine the shape of the wafer parts?

We model the oscillating stripe as a bent block of metal with length l, width w and height h.

Consider an element of the bent stripe. The middle layer ($\lambda = 0$) is not deformed whereas the above layers are strained and the lower layers are pressed.

The torque of those elastic forces equals the torque of the bending force F

$$F(l-x) = \int_{-h/2}^{h/2} \lambda \,\mathrm{d}F_\lambda = \frac{1}{12}\frac{Ewh^3}{r} = \frac{K}{r} = Ky''$$

where E is Young's modulus and r is the local radius of curvature which can be approximated as $1/y''$. After two integrations, we obtain the dependecy of $y(x)$.

$$y(x) = \frac{F}{6K}\left(3lx^2 - x^3\right) .$$

To discover something about dynamics of the bent stripe, we use the law of conservation of energy.

Potential energy of the deformed stripe is

$$E_\mathrm{P} = \int_0^{y(l)} F \,\mathrm{d}y = \int_0^{y(l)} \frac{3K}{l^3} y \,\mathrm{d}y = \frac{3}{2}\frac{K}{l^3} y^2(l) .$$

To get the kinetic energy, we express the velocity of an element in terms of the velocity of the stripe end

$$\dot{y}(x) = \frac{1}{2}\frac{\dot{y}(l)}{l^3}(3x^2 l - x^3) .$$

Total kinetic energy is the sum over all the elements

$$E_\mathrm{K} = \int_0^l \frac{1}{2}\varrho w h \,\mathrm{d}x\, \dot{y}^2(x) = \int_0^l \frac{1}{8}\varrho w h \frac{\dot{y}^2(l)}{l^6}(3x^2 l - x^3)^2 \,\mathrm{d}x = \frac{33}{280}\varrho w h l\, \dot{y}^2(l) .$$

As we know $\mathrm{d}(E_\mathrm{K} + E_\mathrm{P})/\,\mathrm{d}t = 0$, we get the motion equation of harmonic oscillations

$$\frac{33}{140}\varrho w h l\, \ddot{y}(l) + 3\frac{K}{l^3} y(l) = 0 .$$

The frequency of such oscillations is

$$f = \frac{1}{2\pi}\frac{h}{l^2}\sqrt{\frac{35}{33}\frac{E}{\varrho}} .$$

The ratio between any two semitones is constant ($\sqrt[12]{2}$). Hence the relation between the ordinal number (n) of the tone in the sequence and the length of the stripe is $l = l_0 \sqrt[24]{2^{-n}}$.

Michal Koutný
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Fyzikální korespondenční seminář je organizován studenty UK MFF. Je zastřešen Oddělením pro vnější vztahy a propagaci UK MFF a podporován Ústavem teoretické fyziky UK MFF, jeho zaměstnanci a Jednotou českých matematiků a fyziků.